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EQUATIONS OF AIRPLANE MOTION FOR LARGE DISTURBANCES FROM STEADY FLIGHT

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1.0 SUMMARY

Large-disturbance equations of airplane motion suitable for use with modern digital or analog computers are derived from the general equations of rigid-body motion. The procedure followed is to find solutions in each case for the steady-flight conditions about which the desired motions represent perturbations. The variables of motion are expanded to include these steady-flight values, and the steady-flight solutions are applied in order to eliminate the steady-flight forces and moments.

Two specific cases are carried out to final form. These are:

(1) Six degree-of-freedom equations, based on arbitrary body axes, with small longitudinal velocity, sideslip, and angle of attack perturbations.

(2) Five degree-of-freedom (constant longitudinal velocity) equations, based on principal axes, with small sideslip and angle of attack perturbations, and with the effects of gravity simplified.

The first set of equations is shown to be especially suitable for use in fire-control or tracking studies, while the second set of equations is intended for use in calculations involving rapid rolling maneuvers.

Auxiliary equations are developed for the conversion of wind-tunnel data to body-axis stability derivatives and for the iterations required to obtain numerical solutions of the steady-flight equations of motion. Also, as an aid in calculating and interpreting instrument readings in conjunction with large-disturbance motions, the readings of attitude-, velocity-, and acceleration-measuring instruments are derived.

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3.0 INTRODUCTION

The equations of rigid body motion referred to body axes developed by the Swiss mathematician Leonhard Euler are the basis for all airplane dynamic stability theory. These equations have been most widely used in aeronautics in their linearized form, and concepts such as the period and time to damp to half-amplitude of an oscillation, the neutral point, and the transfer function are rigorously defined only in terms of the linearized equations.

Linearization of Euler's equations, as proposed by E. J. Routh, requires that the motions described be restricted to small perturbations about an initial condition of steady motion. An important simplification in the aeronautical field occurs when the initial steady motion is a case of "symmetric" flight, in which the airplane's plane of symmetry remains fixed in a vertical position. Small disturbances from steady symmetric flight are described by two independent sets of three simultaneous equations of motion, instead of by the general single set of six simultaneous equations. In the general linearized case, however, the initial steady motion about which small perturbations occur can be a severe maneuver, such as a steeply banked turn, a rapid roll, or a spiral dive. It is only in cases where the perturbations or disturbances themselves must be allowed to be large that the linearized equations of motion are inapplicable.

The development of certain special non-linear forms of the equations of airplane motion which apply to large disturbances from steady flight is the subject of this report. Specific applications which have been considered are:

(1) Fire-control studies, where the airplane may be required to perform extreme maneuvers while tracking a target

(2) Calculation of rapid rolling maneuvers and similar flight conditions possible with modern airplanes

The inadequacy of the linearized small-perturbation equations of motion for either of these applications has been established. In the expression of the equations of this report in form suitable for machine computation, longitudinal velocity and angle of sideslip and attack disturbances are restricted to small values, as in the linearized theory.

4.0 REFERENCE AXES, SYMBOLS AND DEFINITIONS

4.1 Reference Axes

The large-disturbance equations of airplane motion developed in this report are based on systems of axes which are fixed in the airplane, or body axes. It is often expedient to consider a special case of body axes called principal axes, in which the X-, Y-, and Z-axes coincide with principal axes of inertia. For any set of body axes, however, the following definitions are used. The axes are orthogonal, having their origins at the airplane's center of gravity. The X-axis lies within the airplane's plane of symmetry, and is positive forward. The Z-axis is also in the plane of symmetry, and is positive towards the bottom of the airplane. The Y-axis is perpendicular to both X and Z, and is positive towards the right wing. Positive senses of quantities referred to body axes are governed by the positive directions of the axes themselves, following the right-hand rule in the case of angular velocities and moments. Body axes are illustrated in Figure 1.

Basic aerodynamic data used in the equations of this report are assumed to have been obtained on wind-tunnel stability axes. The wind-tunnel stability axis system is defined identically to airplane body axes except that the Z-stability axis is perpendicular to the relative wind, or approximately to the wind-tunnel axis. In addition, the origin of wind-tunnel stability axes may not coincide with the assumed airplane center of gravity.

4.2 Symbols

Length is measured in feet, mass in slugs, time in seconds, and angles in radians, unless specified otherwise.

b = Wing span

c = Wing chord (Mean Aerodynamic Chord)

C_D, C_L, T'_c = Drag, lift, and thrust coefficients. $C_D = D/Sq$

$$C_L = L/Sq$$

$$T'_c = T/Sq$$

$C_{I_X}, C_{I_Y}, C_{I_Y}', C_{I_Z}, C_{I_{XZ}}$ = Moment of inertia and product of inertia coefficients.

$$C_{I_X} = I_X/Sq_1b$$

$$C_{I_Y} = I_Y/Sq_1c$$

$$C_{I_Y}' = I_Y/Sq_1b$$

$$C_{I_Z} = I_Z/Sq_1b$$

$$C_{I_{XZ}} = I_{XZ}/Sq_1b$$

C_ℓ, C_m, C_n = Rolling, pitching, and yawing moment coefficients.

$$C_\ell = L/Sqb$$

$$C_m = M/Sqc$$

$$C_n = N/Sqb$$

C_X, C_Y, C_Z = Longitudinal, side, and normal force coefficients.

$$C_X = X/Sq$$

$$C_Y = Y/Sq$$

$$C_Z = Z/Sq$$

$$C_{(\ )_{u'}}, C_{(\ )_\beta}, C_{(\ )_\alpha} = \partial C_{(\ )}/\partial u', \partial C_{(\ )}/\partial \beta, \partial C_{(\ )}/\partial \alpha$$

$$C_{(\ )_{\dot\phi}}, C_{(\ )_{\dot\theta}}, C_{(\ )_{\dot\psi}} = \partial C_{(\ )}/\partial p, \partial C_{(\ )}/\partial q, \partial C_{(\ )}/\partial r$$

$$C_{(\ )_p}, C_{(\ )_q}, C_{(\ )_r} = \partial C_{(\ )}/\partial(pb/2V_1), \partial C_{(\ )}/\partial(qc/2V_1), \partial C_{(\ )}/\partial(rb/2V_1)$$

$$C_{(\ )_{\beta\alpha}} = \frac{\partial}{\partial \alpha} C_{(\ )_\beta}, \text{ etc.}$$

d = Accelerometer damping constant

D, L, T = Drag, lift, and net thrust

f = Accelerometer pendulum length

$[G]$ = Gyro matrix

g = Acceleration of gravity

H_e = Engine angular momentum (defined as positive for right-hand rotation, viewed from the rear)

T_V = $\partial T/\partial V$

u, v, w = Disturbance longitudinal, lateral, and normal linear velocities of body axes

U, V, W = Longitudinal, lateral, and normal linear velocities of body axes

u' = Dimensionless small-disturbance velocity variable, u/V_1

V = Total velocity

W = Gross weight of airplane, mg

x, y, z = Distances along X, Y and Z body (airplane) axes

X, Y, Z = Longitudinal, side, and normal aerodynamic forces along body axes

$x_{( )}$, $y_{( )}$, $z_{( )}$ = Body axes stability derivatives

$$x_{( )} = \frac{\partial X}{\partial ( )} \frac{1}{q_1 S}$$

$$y_{( )} = \frac{\partial Y}{\partial ( )} \frac{1}{q_1 S}$$

$$z_{( )} = \frac{\partial Z}{\partial ( )} \frac{1}{q_1 S}$$

z_T = Perpendicular distance of thrust axis below center of gravity

$\underline{i}, \underline{j}, \underline{k}$ = Unit vectors along X, Y and Z body (airplane) axes

I_X, I_Y, I_Z, I_{XZ} = Moments of inertia about the X- Y- and Z- axes, and product of inertia about X and Z.

I = Accelerometer pendulum polar moment of inertia

i_T = Angle of incidence of thrust axis to X- axis.

k = Accelerometer spring constant

$\ell_{(\ )}, m_{(\ )}, n_{(\ )}$ = Body axes stability derivatives

$$\ell_{(\ )} = \frac{\partial L}{\partial (\ )} \frac{1}{q_1 S b}$$

$$m_{(\ )} = \frac{\partial M}{\partial (\ )} \frac{1}{q_1 S c}$$

$$n_{(\ )} = \frac{\partial N}{\partial (\ )} \frac{1}{q_1 S b}$$

$\underline{l}, \underline{m}, \underline{n}$ = Unit vectors along X_I, Y_I, and Z_I instrument axes

L, M, N = Rolling, pitching, and yawing aerodynamic moments

[L] = Orientation matrix

n = Load factor

m = Mass of airplane

m' = Accelerometer seismic mass

p, q, r = Disturbance rolling, pitching, and yawing angular velocities

P, Q, R = Rolling, pitching, and yawing angular velocities of body axes

q = Dynamic pressure, $(\rho/2) V^2$

$\underline{r}$ = Radius vector of instrument location to airplane's center of gravity

s = Operator d/dt

$\underline{s}, \underline{t}, \underline{u}$ = Unit vectors relating to accelerometer (See Figure 7)

S = Wing area

β', α' = Dimensionless small-disturbance velocity variables.

$$\beta' = v/V_1$$

$$\alpha' = w/V_1$$

β, α = Sideslip angle and angle of attack.

$$\beta = \sin^{-1} V/V$$

$$\alpha = \tan^{-1} W/U$$

γ = Flight path angle

δ = Control surface disturbance angle

ϵ = Angle of nose-up incidence of fuselage reference line relative to X - principal axis

ζ = Accelerometer damping ratio

η = Angle of attack of X - principal axis

ρ = Air density, also tilt angle of sensitive direction of accelerometer to the horizontal

τ = Time factor, $m/\rho S V_1$

Ψ, Θ, Φ = Orientation angles of reference axes to arbitrary earth axes

ψ, θ, φ = Disturbance values of the orientation angles

Ω = Total angular velocity

ω_n = Accelerometer undamped natural frequency

Subscripts

a, e, r = Aileron, elevator, rudder

s = Accelerometer sensitive direction

t = Normal to pendulum accelerometer support axis and sensitive directions

u = Pendulum accelerometer support axis direction

I = Instrument or instrument axes

o = Applied force or moment

1 = Steady flight

4.3 Definitions

Steady Flight is defined as flight with zero rates of change of the linear and angular velocity variables, or with $\dot{U}=\dot{V}=\dot{W}=\dot{P}=\dot{Q}=\dot{R}=0$. Steady sideslips and steady level turns or steady helical turns about a vertical space axis are possible steady flight conditions. Steady pitching flight is more properly referred to as "quasi-steady", since $\dot{U}$ or $\dot{W}$ cannot both remain zero over a long period of time if $Q \neq 0$.

Straight Flight is defined as flight with zero values of the angular velocity variables P, Q, and R. Steady sideslips and dives or climbs with or without longitudinal acceleration are some straight flight conditions.

Symmetric Flight is defined as flight with a fixed, vertical, position in space of the plane of symmetry. In symmetric flight, the variables P, R, ϕ, and V remain zero. Wings-laterally-level dives, climbs, pullups and pushdowns, with zero sideslip, are some symmetric flight conditions.

Asymmetric Flight is defined as flight in which the plane of symmetry does not remain in a fixed vertical position. Some or all of the variables of motion P, R, ϕ, or V may be expected to be other than zero in asymmetric flight. Examples of asymmetric flight include sideslips, rolls, and turns.

5.0 DEVELOPMENT OF THE EQUATIONS OF MOTION

5.1 General Equations of Airplane Motion

The general equations of airplane motion are a set of nine simultaneous non-linear differential equations, listed here as Equations (1) to (3):

$$
\left.
\begin{array}{ccc}
\overbrace{\text{Applied Aerodynamic Forces}} & \overbrace{\text{Gravitational Forces}} & \overbrace{\text{Rates of Change of Linear Momentum}} \\
X & -\,mg \sin\theta & = m(\dot{U} - VR + WQ) \\
Y & +\,mg \cos\theta \sin\phi & = m(\dot{V} - WP + UR) \\
Z & +\,mg \cos\theta \cos\phi & = m(\dot{W} - UQ + VP)
\end{array}
\right\} \quad (1)
$$

$$
\left.
\begin{array}{cccc}
\overbrace{\text{Applied Aerodynamic Moments}} & & \overbrace{\text{Rates of Change of Angular Momentum, Engine Stopped}} & \overbrace{\text{Rates of Change of Engine Angular Momentum}} \\
L & = & I_X\dot{P} - I_{XZ}(\dot{R}+PQ) + (I_Z - I_Y)QR & -\,QH_e \sin i_T \\
M & = & I_Y\dot{Q} + I_{XZ}(P^2 - R^2) + (I_X - I_Z)PR & +\,PH_e \sin i_T + RH_e \cos i_T \\
N & = & I_Z\dot{R} + I_{XZ}(-\dot{P} + QR) + (I_Y - I_X)PQ & -\,QH_e \cos i_T
\end{array}
\right\} (2)
$$

$$
\left.
\begin{array}{ccc}
\overbrace{\text{Rates of Change of Orientation Angles}} & & \overbrace{\text{Functions of Angular Velocities}} \\
\dot{\phi} & = & P + (Q \sin\phi + R \cos\phi) \tan\theta \\
\dot{\theta} & = & Q \cos\phi - R \sin\phi \\
\dot{\psi} & = & (Q \sin\phi + R \cos\phi) \sec\theta
\end{array}
\right\} (3)
$$

Equations (2) apply to airplanes having mirror symmetry about the X-Z plane except for engine angular momentum. Consequently, the effects of rudder and aileron deflections on the inertia parameters are neglected. The aerodynamic forces and moments in Equations (1) and (2) are usually found to depend upon the linear and angular velocities and their derivatives, the control surface angles and their derivatives, air density, Mach number, and possibly other factors as well. These functional dependences are represented by Equation (4).

$$\left.\begin{aligned}
X &= X\,(U, V, W, \dot U, \dot V, \dot W, \ldots\ldots\ldots P, Q, R, \dot P, \dot Q, \dot R, \ldots\ldots\ldots\ldots \\
&\qquad \delta_a, \delta_e, \delta_r, \dot\delta_a, \dot\delta_e, \dot\delta_r, \ldots\ldots\ldots, \beta, M, \ldots\ldots\ldots) \\
Y &= Y\,(\qquad\qquad " \qquad\qquad) \\
Z &= Z\,(\qquad\qquad " \qquad\qquad) \\
L &= L\,(\qquad\qquad " \qquad\qquad) \\
M &= M\,(\qquad\qquad " \qquad\qquad) \\
N &= N\,(\qquad\qquad " \qquad\qquad)
\end{aligned}\right\} (4)$$

It is usual to find that much simpler functions than implied by Equation (4) are adequate in engineering problems. In addition, the variable set U, V, W is generally replaced by the equivalent set $\mathbf{V}$, β, α, where:

$$\left.\begin{aligned}
U &= \mathbf{V} \cos\alpha \cos\beta \\
V &= \mathbf{V} \sin\beta \\
W &= \mathbf{V} \sin\alpha \cos\beta
\end{aligned}\right\} (5)$$

alternately:

$$\left.\begin{aligned}
\mathbf{V} &= \sqrt{U^2 + V^2 + W^2} \\
\beta &= \sin^{-1}(V/\mathbf{V}) \\
\alpha &= \tan^{-1}(W/U)
\end{aligned}\right\} (5A)$$

Derivations of Equations (1) and (2), with the exception of the engine momentum terms, may be found in Reference 1. The engine momentum terms in Equation (2) are derived in Reference 2. Derivations of Equations (3) and (5) are given in Reference 3. Dimensionless forms of Equations (1) and (2) are obtained by dividing by q_1S and by q_1Sb or q_1Sc respectively:

$$\left.\begin{aligned}
\frac{X}{q_1S} - \frac{2\tau g}{\mathbf{V}_1} \sin\theta &= \frac{2\tau}{\mathbf{V}_1}(\dot U - VR + WQ) \\
\frac{Y}{q_1S} + \frac{2\tau g}{\mathbf{V}_1} \cos\theta \sin\phi &= \frac{2\tau}{\mathbf{V}_1}(\dot V - WP + UR) \\
\frac{Z}{q_1S} + \frac{2\tau g}{\mathbf{V}_1} \cos\theta \cos\phi &= \frac{2\tau}{\mathbf{V}_1}(\dot W - UQ + VP)
\end{aligned}\right\} (1A)$$

$$\left.\begin{aligned}
\frac{L}{q_1Sb} &= C_{I_X}\dot P - C_{I_{XZ}}(\dot R + PQ) + (C_{I_Z} - C'_{I_Y})QR - \frac{H_e \sin i_T}{Sq_1b}Q \\
\frac{M}{q_1Sc} &= C_{I_Y}\dot Q + C_{I_{XZ}}\frac{b}{\bar c}(P^2 - R^2) + (C_{I_X} - C_{I_Z})\frac{b}{c}PR + \frac{H_e \sin i_T}{Sq_1c}P + \frac{H_e \cos i_T}{Sq_1c}R \\
\frac{N}{q_1Sb} &= C_{I_Z}\dot R + C_{I_{XZ}}(-\dot P + QR) + (C'_{I_Y} - C_{I_Y})PQ - \frac{H_e \cos i_T}{Sq_1b}Q
\end{aligned}\right\} (2A)$$

5.2 Steady-Flight Equations of Motion

The steady-flight case of the general equations of airplane motion is useful for establishing initial conditions for the numerical solution of the general equations, as well as for calculations involving steady flight only. In the most general expression the steady-flight equations of airplane motion are obtained from Equations (1) and (2) by eliminating the terms involving derivatives of the variables (U,V,W,P,Q,R), and employing the steady-flight subscript 1 throughout.

Two special cases of the steady-flight equations of airplane motion are of particular interest, and these cases are discussed in the following sections.

5.2.1 Steady Pitching Symmetric Flight

Rolling-pullout or pushdown maneuvers may be performed by applying abrupt aileron deflections while in steady symmetric pullup or pushdowns. Using the dimensionless forms of Equations (1) and (2), with α obtained from Equation (5), the steady-flight equations which apply are:

$$\left.\begin{aligned} &C_{X_1} - \frac{2\tau g}{V_1}\sin(\gamma_1+\alpha_1) = 2\tau Q_1 \sin\alpha_1 \\ &C_{Y_1} = 0 \\ &C_{Z_1} + \frac{2\tau g}{V_1}\cos(\gamma_1+\alpha_1) = -2\tau Q_1 \cos\alpha_1 \end{aligned}\right\} \quad (6)$$

$$\left.\begin{aligned} &C_{l_1} = -\frac{H_e \sin i_T}{S q_1 b} Q_1 \\ &C_{m_1} = 0 \\ &C_{n_1} = -\frac{H_e \cos i_T}{S q_1 b} Q_1 \end{aligned}\right\} \quad (7)$$

The steady pitching velocity Q_1 is often expressed in terms of the normal load factor n_Z. Load factor is defined as the ratio of applied aerodynamic force in the specified direction to the gross weight. Thus:

$$n_Z = -\frac{Z}{W} \qquad (8)$$

The negative sign in Equation (8) is used to obtain agreement with the usual sign convention. Substituting Equation (8) into the third of Equations (6) and solving for Q_1,

$$Q_1 = \frac{g}{V_1 \cos\alpha_1}\left[n_{Z_1} - \cos(\gamma_1 + \alpha_1)\right] \qquad (9)$$

The numerical solutions of Equations (6), (7) and (9) are discussed in Section 8.1. Because of the arbitrary functional dependence of C_{X_1} and C_{Z_1} on angle of attack, this solution is found by iteration.

5.2.2 Steady Straight Symmetric Flight

Steady straight symmetric flight may be used as a reference for studying almost any maneuver, using the large-disturbance equations of airplane motion. However, because of practical limitations in the way of information storage on either digital or analog computers, it will be found expedient to establish initial or reference conditions which are as close to the desired maneuver as is possible. For example, although steady straight symmetric flight may be used as a reference for studying the rolling pullout maneuver, initial conditions which include the steady pitching portion of the maneuver (as in 5.2.1) eliminate the necessity for machine computation of that much of the problem.

Where steady straight symmetric flight is chosen as the initial steady flight condition, the applicable equations of motion are Equations (6) and (7), with $Q_1 = 0$.

5.3 Expansions of the Variables to Include Initial Conditions

The steady-flight portions of the general equations of airplane motion can be eliminated from the equations. As previously indicated, this will reduce the amount of information storage needed for numerical solutions of the equations. The steady-flight portion of the equations may be eliminated if the variables of the motion are expanded to include initial conditions. This is done in the following equations for the attitude angles, the angular velocities and accelerations, and the linear velocities and accelerations. It should be appreciated that in these equations the disturbance quantities indicated by lower-case symbols or by the prefix Δ are not necessarily small disturbances, as they are in Reference 4.

$$\theta = \theta_1 + \vartheta \tag{10}$$

and

$$\left.\begin{aligned} \sin\theta &= \sin\theta_1 \cos\vartheta + \cos\theta_1 \sin\vartheta \\ \cos\theta &= \cos\theta_1 \cos\vartheta - \sin\theta_1 \sin\vartheta \end{aligned}\right\} \tag{11}$$

$$\phi = \phi_1 + \varphi \tag{12}$$

and

$$\left.\begin{aligned} \sin\Phi &= \sin\Phi_1\cos\varphi + \cos\Phi_1\sin\varphi \\ \cos\Phi &= \cos\Phi_1\cos\varphi - \sin\Phi_1\sin\varphi \end{aligned}\right\}\quad(13)$$

$$\left.\begin{aligned} P &= P_1 + p \\ Q &= Q_1 + q \\ R &= R_1 + r \end{aligned}\right\}\quad(14)$$

$$\left.\begin{aligned} \dot{P} &= \dot{p} \\ \dot{Q} &= \dot{q} \\ \dot{R} &= \dot{r} \end{aligned}\right\}\quad(15)$$

$$\left.\begin{aligned} \beta &= \beta_1 + \Delta\beta = \Delta\beta \\ \alpha &= \alpha_1 + \Delta\alpha \end{aligned}\right\}\quad(16)^*$$

$$\left.\begin{aligned} \dot{\beta} &= \Delta\dot{\beta} \\ \dot{\alpha} &= \Delta\dot{\alpha} \end{aligned}\right\}\quad(17)$$

$$\mathbf{V} = \mathbf{V}_1 + \Delta\mathbf{V}\quad(18)$$

$$\left.\begin{aligned} U &= U_1 + u = \mathbf{V}\cos\beta\cos\alpha \\ &= (\mathbf{V}_1 + \Delta\mathbf{V})\cos\Delta\beta\,(\cos\alpha_1\cos\Delta\alpha - \sin\alpha_1\sin\Delta\alpha) \\ V &= V_1 + v = \mathbf{V}\sin\beta \\ &= (\mathbf{V}_1 + \Delta\mathbf{V})\sin\Delta\beta \\ W &= W_1 + w = \mathbf{V}\cos\beta\sin\alpha \\ &= (\mathbf{V}_1 + \Delta\mathbf{V})\cos\Delta\beta\,(\sin\alpha_1\cos\Delta\alpha + \cos\alpha_1\sin\Delta\alpha) \end{aligned}\right\}\quad(19)$$

* The steady-state sideslip β_1 is set = 0, as suggested in Section 6.0.

$$\left.\begin{aligned}
\dot{U} = \dot{u} = &-(V_1 + \Delta V) \cos \Delta\beta \,(\sin \alpha_1 \cos \Delta\alpha + \cos \alpha_1 \sin\Delta\alpha)\Delta\dot{\alpha} \\
&-(V_1 + \Delta V) \sin \Delta\beta \,(\cos \alpha_1 \cos \Delta\alpha - \sin \alpha_1 \sin\Delta\alpha)\Delta\dot{\beta} \\
&+ \Delta\dot{V} \cos \Delta\beta \,(\cos \alpha_1 \cos \Delta\alpha - \sin \alpha_1 \sin \Delta\alpha) \\
\dot{V} = \dot{v} = &(V_1 + \Delta V) \cos \Delta\beta \,\Delta\dot{\beta} + \Delta\dot{V} \sin \Delta\beta \\
\dot{W} = \dot{w} = &(V_1 + \Delta V) \cos \Delta\beta \,(\cos \alpha_1 \cos \Delta\alpha - \sin \alpha_1 \sin \Delta\alpha)\Delta\dot{\alpha} \\
&-(V_1 + \Delta V) \sin \Delta\beta \,(\sin \alpha_1 \cos \Delta\alpha + \cos \alpha_1 \sin \Delta\alpha)\Delta\dot{\beta} \\
&+ \Delta\dot{V} \cos \Delta\beta \,(\sin \alpha_1 \cos \Delta\alpha + \cos \alpha_1 \sin \Delta\alpha)
\end{aligned}\right\} \quad (20)$$

Equations (18) to (20) express linear velocities and accelerations along airplane body axes in terms of the large-disturbance variables ΔV, $\Delta\beta$, and $\Delta\alpha$. Aerodynamic flow breakdown always acts to limit the practical ranges of these variables, particularly the latter two. This makes it expedient to use the small-disturbance linear velocity variables u', β', and α' in place of ΔV, $\Delta\beta$, and $\Delta\alpha$ respectively, even though the angular displacement and velocity disturbance variables ψ, θ, φ, p, q, and r are allowed to take on large values. The small-disturbance linear velocity variables are defined as:

$$\left.\begin{aligned}
u' &= \frac{u}{V_1} \\
\beta' &= \frac{v}{V_1} \\
\alpha' &= \frac{w}{V_1}
\end{aligned}\right\} \quad (21)$$

Following R.M. Jones on page 146 of Reference 4, the total velocity change ΔV is expressed in terms of u' and α' as follows:

From Figure 3:

$$V^2 = U^2 + V^2 + W^2 \quad (22)$$

Differentiating:

$$2V_1\, dV = 2U_1\, u + 2V_1\, v + 2W_1 w$$

Since we assume that $V_1 = 0$:

$$\frac{\Delta V}{V_1} = u' \cos \alpha_1 + \alpha' \sin \alpha_1 \quad (23)$$

From Equation (19)

$$\Delta\beta = \sin^{-1} \frac{v}{V_1 + \Delta V}$$

Since both v and ΔV are small the sine equals the angle and ΔV is negligible compared with V_1. Thus, from Equations (21)

$$\Delta\beta = \beta' \tag{24}$$

From Equations (16) and (19):

$$\Delta\alpha = \alpha - \alpha_1$$

$$= \tan^{-1}\frac{W_1+w}{U_1+u} - \tan^{-1}\frac{W_1}{U_1}$$

$$= \tan^{-1}\frac{\dfrac{W_1+w}{U_1+u} - \dfrac{W_1}{U_1}}{1 + \dfrac{(W_1+w)\,W_1}{(U_1+u)\,U_1}}$$

Simplifying and substituting V_1^2 for $U_1^{\,2} + W_1^{\,2}$,

$\cos\alpha_1$ for U_1/V_1, and $\sin\alpha_1$ for W_1/V_1,

$$\Delta\alpha = \tan^{-1}\frac{w\cos\alpha_1 - u\sin\alpha_1}{V_1 + u\cos\alpha_1 + w\sin\alpha_1}$$

w and u are both assumed small, and negligible compared with V_1. Thus, from Equations (21):

$$\Delta\alpha = \alpha'\cos\alpha_1 - u'\sin\alpha_1 \tag{25}$$

Note finally that since β' and α' are small,

$$\left.\begin{array}{ll} \sin\beta' = \beta' & \cos\beta' = 1.0 \\ \sin\alpha' = \alpha' & \cos\alpha' = 1.0 \end{array}\right\} \tag{26}$$

and products of u', β', α', and their derivatives are negligible.

Equations (23) to (26) may now be substituted into Equations (19) and (20), leading to:

$$\left.\begin{array}{l} U = V_1\,(u' + \cos\alpha_1) \\ V = V_1\,\beta' \\ W = V_1\,(\alpha' + \sin\alpha_1) \end{array}\right\} \tag{27}$$

$$\left.\begin{array}{l} \dot{U} = V_1\,\dot{u}' \\ \dot{V} = V_1\,\dot{\beta}' \\ \dot{W} = V_1\,\dot{\alpha}' \end{array}\right\} \tag{28}$$

Equations (27) and (28) can of course be obtained directly from Equations (19) and (21) using these relationships from Figure 3 (with $V_1 = 0$):

$$\left.\begin{aligned} \cos \alpha_1 &= U_1/V_1 \\ \sin \alpha_1 &= W_1/V_1 \end{aligned}\right\} \quad (29)$$

In order to obtain numerical solutions of the equations of airplane motion the aerodynamic forces and moments must be expressed in terms of the linear and angular velocity variables of motion. In the general case of large disturbances, these expressions, symbolized in Equations (4), can be exceedingly non-linear and complex. These difficulties arise from aerodynamic flow separation at large relative flow angles and at high subsonic Mach numbers, from the interacting effects of large values of the variables like sideslip and angle of attack, and from possible aerodynamic hysteresis effects. Where aerodynamic non-linearities can be disregarded, Equations (30) are illustrative of the expansion of the aerodynamic forces and moments. These equations are made dimensionless to conform with Equations (1A) and (2A), and the small-disturbance linear velocity variables are employed.

$$\left.\begin{aligned} \frac{X}{q_1 S} &= C_{X_1} + \frac{1}{q_1 S}\left(\frac{\partial X}{\partial u'} u' + \frac{\partial X}{\partial \alpha} \alpha' + \ldots\ldots\right) \\ \frac{Y}{q_1 S} &= C_{Y_1} + \ldots\ldots\ldots \text{ etc.} \\ &\text{etc.} \end{aligned}\right\} \quad (30)$$

6.0 SPECIAL FORMS OF THE EQUATIONS OF MOTION

The general equations of airplane motion for arbitrary body axes are given with reasonable generality in Equations (1) to (5). Several special forms of these equations are developed in this section, making use of a series of approximations which are designed to simplify the equations as far as possible, consistent with their intended uses.

6.1 Equations Suitable for Tracking Studies

In studying the closed-loop dynamics of an automatic tracking or fire-control loop it is typical to calculate the motions of the attacking airplane from the initiation of automatic tracking to the firing point. The initial conditions are characterized by flight at zero or small bank angles. During the maneuver, large bank angle changes may be expected, together with airspeed changes that tend to increase as the length of the maneuver is prolonged.

Accordingly, the initial conditions selected for this type of study are steady straight symmetric flight (see Definitions and Section 5.2.2), where the following quantities are zero:

$\beta_1 \; P_1 \; Q_1 \; R_1 \; \Phi_1$

In the expansions used in these equations of motion the small-disturbance linear velocity variables are used and the aerodynamic forces and moments are linearized as in Equation (30). It may be appreciated that in particular cases, non-linear aerodynamic force and moment characteristics may be inserted into these equations.

Without repeating the lengthy algebraic manipulations involved, the development of the tracking equations of motion may be outlined in these steps:

1. Substitute the velocity, angular position and aerodynamic force and moment expansion equations (10) to (15), (27), (28), and (30) into Equations (1A) and (2A).
2. Cancel the initial aerodynamic force and moment coefficients, using Equations (6) and (7).

The final equations of motion are shown in matrix form in Table I. The column headings of Table I are the variables of motion and functions such as the products of the variables of motion. The rows represent separate, independent equations of motion. The tabular entries are coefficients of the variables or functions of the variables which appear in the particular equation. Thus, the first row of Table I is interpreted as:

$$\left(\frac{x_{u'}}{2\tau} - \gamma_1\right) u' + \frac{x_{\alpha'}}{2\tau} \alpha' - \sin \alpha_1 Q + \ldots\ldots + \frac{x_{\delta_e}}{2\tau} \delta_e = 0$$

If principal axes are chosen as the arbitrary body axes of Table I, $I_{XZ} = 0$ and $\alpha_1 = \eta_1$.

6.2 Equations Suitable for the Calculation of Rapid Rolling Maneuvers

Modern high-speed airplanes are capable of high rolling velocities starting from either initially straight flight or from pullups or pushdowns. The inadequacy of the classical small-disturbance equations of airplane motion to calculate properly these motions has become well known.

These significant portions of rapid rolling maneuvers generally are of short duration after the initiation of the roll, and airspeed changes are consequently small and of negligible influence on the motion. Furthermore, in conditions where non-linear inertial couplings play an important part in the motions and the classical linearized equations of motion are least applicable, aerodynamic and gravity effects are correspondingly less significant and more susceptible to drastic simplification.

The initial conditions selected for this type of study are steady pitching or straight symmetric flight (see Definitions and Sections 5.2.1 and 5.2.2), where the following quantities are zero: β_1, P_1, R_1, ϕ_1 . The small-disturbance linear velocity variables are used and the aerodynamic forces and moments are linearized as in Equations (30). Also, constant longitudinal velocity ($u = \dot{u} = 0$), and zero pitch angle, as it affects the force due to gravity ($\theta_1 = \theta = 0$) are assumed.

The final equations of motion for this case are derived very much as outlined in the previous section, and these equations are shown in matrix form in Table II. Principal axes are chosen as the arbitrary body axes, in order to reduce the required multiplications of variables. This appears to be a reasonable simplification for practical computations, since there is no tracking control data system with arbitrary body axis orientations to be supplied with computed airplane motions, as in the first example.

7.0 STABILITY DERIVATIVES ON BODY AXES

The body axes stability derivatives appearing in Tables I and II are related to wind-tunnel data referred to stability axes in this section. It is assumed that moment transfers to the assumed airplane center of gravity have already been performed on the wind-tunnel data, if needed.

7.1 Longitudinal Velocity Derivatives

The steps involved in deriving the longitudinal velocity derivatives on body axes from wind-tunnel data are somewhat tedious, and they will be fully illustrated in only one case. The illustrative example is the body axes derivative $x_{u'}$, defined as $\frac{\partial X}{\partial u'} \frac{1}{q_1 S}$. The X force may be written as:

$$X = C_X (\rho/2) V^2 S \qquad (31)$$

Differentiating

$$\frac{\partial X}{\partial U} = \frac{\rho}{2} V_1^2 S \frac{\partial C_X}{\partial U} + C_{X_1} \left(\frac{\rho}{2}\right) S \, 2V_1 \frac{\partial V}{\partial U}$$

Now, $u' = U/V_1$ and $\partial V/\partial U = \cos \alpha_1$ from page 146 of Reference 4. Then:

$$\frac{\partial X}{\partial u'} \frac{1}{q_1 S} = C_{X_{u'}} + 2 C_{X_1} \cos \alpha_1 \qquad (32)$$

The derivative $C_{X_{u'}}$ can have contributions due to compressibility and aeroelastic effects in addition to the effects of angle of attack treated in Reference 4. Thus:

$$C_{X_{u'}} = C_{X_\alpha} \frac{\partial \alpha}{\partial u'} + C_{X_M} \frac{\partial M}{\partial u'} + C_{X_q} \frac{\partial q}{\partial u'} \qquad (33)$$

From page 146 of reference 4:

$$\frac{\partial \alpha}{\partial U} = - \frac{\sin \alpha_1}{V_1}$$

and

$$\frac{\partial \alpha}{\partial u'} = - \sin \alpha_1 \qquad (34)$$

The partial derivatives $\partial M/\partial u'$ and $\partial q/\partial u'$ for arbitrary body axes are not treated in Reference 4, so these are derived in the following steps:

$$M = \frac{V}{a} = \frac{V_1 + dV}{a}$$

$$= M_1 \left(1 + \frac{dV}{V_1}\right) \qquad (35)$$

The differential dV is obtained from the total velocity equation:

$$V^2 = U^2 + V^2 + W^2$$

Differentiating:

$$2V_1\, dV = 2U_1 dU + 2V_1 dV + 2W_1 dW$$

With the initial lateral velocity V_1 equal to zero and with $u' = dU/V_1$, $\alpha' = dW/V_1$, $U_1/V_1 = \cos \alpha_1$, and $W/V_1 = \sin \alpha_1$:

$$\frac{dV}{V_1} = u' \cos \alpha_1 + \alpha' \sin \alpha_1 \qquad (36)$$

Substituting Equation (36) into Equation (35), and taking partials:

$$\left.\begin{aligned} \frac{\partial M}{\partial u'} &= M_1 \cos \alpha_1 \\ \frac{\partial M}{\partial \alpha'} &= M_1 \sin \alpha_1 \end{aligned}\right\} \quad (37)$$

Again:

$$V = V_1 \left(1 + \frac{dV}{V_1}\right)$$

From Equation (36), using $q = (\rho/2) V^2$, and neglecting products of u' and α':

$$q = \frac{\rho}{2} V_1^{\,2} (1 + 2u' \cos \alpha_1 + 2\, \alpha' \sin \alpha_1) \qquad (38)$$

Taking partial derivatives:

$$\left.\begin{aligned} \frac{\partial q}{\partial u'} &= \rho V_1^{\,2} \cos \alpha_1 \\ \frac{\partial q}{\partial \alpha'} &= \rho V_1^{\,2} \sin \alpha_1 \end{aligned}\right\} \quad (39)$$

In the next part of this derivation, the longitudinal and normal aerodynamic force coefficients C_X and C_Z are related to the force coefficients measured on wind-tunnel stability axes C_L, C_D, and to the calculated net thrust coefficient T_c'. It is assumed that the effects of running jet engines are all represented with sufficient accuracy in the wind-tunnel tests except for the net thrust. That is, it is assumed that the intake normal force discussed in Reference 5 and any air intake drag forces are included in the forces and moments measured in the wind-tunnel. This will be the case when the full-scale jet inlet velocity ratio is matched in the wind tunnel, and the duct inlet details are represented. With these conditions, C_X and C_Z are obtained from Figure 5 as:

$$\left.\begin{aligned} C_X &= C_L \sin\alpha - C_D \cos\alpha + T_c' \cos i_T \\ C_Z &= -C_L \cos\alpha - C_D \sin\alpha - T_c' \sin i_T \end{aligned}\right\} \quad (40)$$

Equations (40) can be differentiated, yielding for the X - partial derivatives of Equation (33):

$$\begin{aligned} C_{X_\alpha} &= (C_{L_\alpha} + C_{D_1}) \sin\alpha_1 + (C_{L_1} - C_{D_\alpha}) \cos\alpha_1 \\ C_{X_M} &= C_{L_M} \sin\alpha_1 - C_{D_M} \cos\alpha_1 + T_c{}'_M \cos i_T \\ C_{X_q} &= C_{L_q} \sin\alpha_1 - C_{D_q} \cos\alpha_1 \end{aligned} \quad (41)$$

The variation of thrust with airspeed represented by the partial derivative $T_c{}'_M$ is more conveniently expressed in terms of the rate of change of net thrust with airspeed T_V. This is accomplished in the following steps. Let

$$T = T_c' (\rho/2) V^2 S$$

Then, as in the derivation of Equation (32),

$$\frac{\partial T}{\partial u'} \frac{1}{q_1 S} = T_c{}'_{u'} + 2T_c{}'_1 \cos\alpha_1 \quad (42)$$

Now:

$$\frac{\partial T}{\partial V} = \frac{\partial T}{\partial u'} \frac{1}{V_1} \frac{\partial U}{\partial V} \quad (43)$$

and $\partial U/\partial V = 1/\cos\alpha_1$, from Reference 4. Also:

$$T_c{}'_{u'} = \frac{\partial T_c'}{\partial M} \frac{\partial M}{\partial u'} = T_c{}'_M M_1 \cos\alpha_1 \quad (44)$$

Substituting Equations (43) and (44) into Equation (42),

$$T_c{}'_M = T_V \frac{a}{q_1 S} - \frac{2T_c{}'_1}{M_1} \quad (45)$$

The final step required is the substitution of Equations (33), (34), (37), (39), (40), (41), and (45) into Equation (32), and simplifying. When this procedure has been carried out, the results may be summarized as on the first row of Table III. The other longitudinal velocity derivatives on body axes are obtained in like manner.

7.2 Angle of Attack Derivatives

The derivation of the angle of attack derivatives on body axes is illustrated in a single case, as for the longitudinal velocity derivatives. The illustrative example is $x_{\alpha'}$, defined as $\frac{\partial X}{\partial \alpha'} \frac{1}{q_1 S}$. Differentiating Equation (31) with respect to W, and noting that $\alpha' = \frac{W}{V_1}$ and $\partial V/\partial W = \sin \alpha_1$ from page 146 of Reference 4,

$$\frac{\partial X}{\partial \alpha'} \frac{1}{q_1 S} = C_{X_{\alpha'}} + 2C_{X_1} \sin \alpha_1 \tag{46}$$

Now:

$$C_{X_{\alpha'}} = C_{X_\alpha} \frac{\partial \alpha}{\partial \alpha'} + C_{X_M} \frac{\partial M}{\partial \alpha'} + C_{X_q} \frac{\partial q}{\partial \alpha'} \tag{47}$$

From page 146 of Reference 4:

$$\frac{\partial \alpha}{\partial W} = \frac{\cos \alpha_1}{V_1}$$

and

$$\frac{\partial \alpha}{\partial \alpha'} = \cos \alpha_1 \tag{48}$$

Finally, by substituting Equations (37), (39), (40), (41), (45), (47), and (48) into Equation (46), the derivative $x_{\alpha'}$ is obtained as the first row of Table IV. The remaining angle of attack derivatives on body axes are obtained in like manner.

7.3 Sideslip Derivatives

The relationships between the sideslip derivatives on arbitrary body axes and the sideslip derivatives on stability axes obtained in the wind tunnel are derived on page 61 of Reference 4. These equations are reproduced in Table V of this report, using modern notation. Second-order derivatives like $y_{\beta'\alpha'}$ are included in the equations of Tables I and II, having the significance of the rate of change with angle of attack of the sideslip derivative in question. Formulae for obtaining the second-order sideslip derivatives directly from wind-tunnel data are included in Table V.

7.4 Symmetric Rotary Derivatives

The derivation of the symmetric rotary derivatives on body axes in terms of wind-tunnel or calculated data on stability axes is illustrated by the derivation of $x_{\dot\theta}$, defined as $\frac{\partial X}{\partial q} \frac{1}{q_1 S}$.

Differentiating Equation (31) with respect to q and noting that $\partial V/\partial q = 0$,

$$\frac{\partial X}{\partial q} \frac{1}{q_1 S} = C_{X_{\dot\theta}} \tag{49}$$

The disturbance pitching velocity variable q is independen. of angle of attack, Mach number, dynamic pressure, and the other variables of the problem. Thus, differentiating Equation (40) with respect to q,

$$C_{X_{\dot\theta}} = C_{L_{\dot\theta}} \; s \qquad {}_1 - C_{D_{\dot\theta}} \cos \alpha_1 \tag{50}$$

The other symmetric rotary derivatives summarized in Table VI are derived in like manner.

7.5 Asymmetric Rotary Derivatives

The relationships between the rolling and ya· ·g or asymmetric rotary derivatives on arbitrary body axes ar. these same derivatives referred to stability axes are .erived on pages 71 and 72 of Reference 4. These expressions are complicated by the nee. to resolve not only the forces and moments on the body axes, but also the rolling and yawing velocities. The results of these derivations are summarized in Table VII, including the second-order asymmetric rotary derivatives.

7.6 Control Surface Derivatives

The control surface derivatives .n body axes in terms of wind-tunnel data on stability axes may be derived in a similar manner to the symmetric rotary derivatives in Section 7.4. This is the .nsequence of the independence of the control surface disturbance angles δ with respect to the other variables of the motion. The body axes control surface derivatives are shown in Table VIII. Second-order control surface derivatives like $x_{\delta\alpha'} = \frac{\partial}{\partial \alpha'} \left(\frac{\partial X}{\partial \delta}\right) \frac{1}{q_1 S}$ are not shown in Table I, II, or VIII, but these can be readily included if needed.

8.0 DEVELOPMENT OF AUXILIARY EQUATIONS

A certain number of auxiliary equations must be developed in order to apply properly the large-disturbance equations of motion presented in Tables I and II. These equations are developed in this portion of the report, followed by the relations needed to calculate the indications of flight instruments from the computed motions.

8.1 Initial Condition Iterations

The steady-flight equations of motion (6) and (7) are used as initial conditions for the equations of Table I and II. As pointed out in Section 5.2.1 numerical solutions of these steady-flight equations of motion normally require iteration because of the arbitrary functional dependence of C_{X_1} and C_{Z_1} on angle of attack. A suggested procedure for this solution is outlined in this section. It is assumed that the airspeed, altitude, flight path angle, gross weight, and normal load factor initial flight conditions are specified. The suggested iteration procedure follows:

(1) Select a trial value of angle of attack, and find the corresponding values of C_{L_1} and C_{D_1} from wind-tunnel data.

(2) Solve for Q_1 from Equation (9).

(3) Using the previous values for α_1, C_{L_1}, C_{D_1}, and Q_1, solve for T_{c_1}' from Equations (51) and (52). These equations were obtained by the substitution of Equations (40) into Equations (6).

$$T_{c_1}' = \frac{1}{\cos i_T}\left[-C_{L_1}\sin\alpha_1 + C_{D_1}\cos\alpha_1 + \frac{2\Upsilon g}{V_1}\sin(\gamma_1+\alpha_1) + 2\Upsilon Q_1\sin\alpha_1\right] \quad (51)$$

$$T_{c_1}' = \frac{1}{\sin i_T}\left[-C_{L_1}\cos\alpha_1 - C_{D_1}\sin\alpha_1 + \frac{2\Upsilon g}{V_1}\cos(\gamma_1+\alpha_1) + 2\Upsilon Q_1\cos\alpha_1\right] \quad (52)$$

(4) Plotting the trial solutions for T_{c_1}' from Equations (51) and (52) against the trial values of α_1 will always lead to a rapid convergence.

8.2 Readings of Attitude - Measuring Instruments

The general expression for computing the readings of the attitude-measuring instruments in a maneuvering airplane may be concisely stated in the notation of Reference (3) as:

$$[U] = [L] \quad (53)$$

where $[G]$ is a matrix representing the successive non-interacting rotations performed by the case of the measuring instrument to arrive at the airplane's orientation and $[L]$ is the orientation matrix of Reference 3. If the attitude-measuring instruments are conventional vertical and directional free gyros, Equation (53) reduces to Equations (14) and (16) of Reference 3.

8.3 Readings of Velocity - Measuring Instruments

In the most general case, velocity-measuring instruments may have any arbitrary location and orientation with relation to a maneuvering airplane. Of course, unless the velocity-measuring instruments are located at the airplane's center of gravity and are aligned with the arbitrary system of body axes, the instrument readings (corrected for local-flow distortion position errors such as upwash or sidewash, instrument errors and reduced to true speed) will not be given by U, V, and W of Equations (5), (5A), or (19).

A general set of instrument axes may be selected, with origin at a radius vector $\underline{r}$ from the airplane's center of gravity, and with arbitrary orientation angles ψ_I, θ_I, and ϕ_I with respect to the body axis system. These general instrument axes are illustrated in Figure 6. The true speed on instrument axes may be expressed as:

$$\underline{V}_I = \underline{V} + \underline{\Omega} \times \underline{r} \qquad (54)$$

where

$$\underline{V}_I = \underline{\ell}\, U_I + \underline{m}\, V_I + \underline{n}\, W_I$$

$$\underline{V} = \underline{i}\, U + \underline{j}\, V + \underline{k}\, W$$

$$\underline{\Omega} = \underline{i}\, P + \underline{j}\, Q + \underline{k}\, R$$

$$\underline{r} = \underline{i}\, x + \underline{j}\, y + \underline{k}\, z$$

Equation (54) may be expressed in Cartesian form using the Orientation Matrix $[L]$ of Reference 3, but with ψ_I, θ_I, and ϕ_I in place of ψ, θ, and ϕ. Thus:

$$\begin{bmatrix} U_I \\ V_I \\ W_I \end{bmatrix} = \begin{bmatrix} L_I \end{bmatrix} \begin{bmatrix} U - Ry + Qz \\ V - Pz + Rx \\ W - Qx + Py \end{bmatrix} \qquad (55)$$

or

$$\begin{aligned} U_I = & (U - Ry + Qz)\cos\theta_I\cos\psi_I \\ & + (V - Pz + Rx)\cos\theta_I\sin\psi_I \\ & - (W - Qx + Py)\sin\theta_I \end{aligned}$$

$$\begin{aligned} V_I = & (U - Ry + Qz)(\cos\psi_I\sin\phi_I\sin\theta_I - \sin\psi_I\cos\phi_I) \\ & + (V - Pz + Rx)(\sin\psi_I\sin\phi_I\sin\theta_I + \cos\psi_I\cos\phi_I) \\ & + (W - Qx + Py)\sin\phi_I\cos\theta_I \end{aligned}$$

$$\begin{aligned} W_I = & (U - Ry + Qz)(\cos\psi_I\cos\phi_I\sin\theta_I + \sin\psi_I\sin\phi_I) \\ & + (V - Pz + Rx)(\sin\psi_I\cos\phi_I\sin\theta_I - \cos\psi_I\sin\phi_I) \\ & + (W - Qx + Py)\cos\theta_I\cos\phi_I \end{aligned}$$

In the usual flight instrument arrangement where sideslip and angle of attack vanes are provided which are independent of one another, only the angle of attack signal is consistent with the conventions of Figure 3. For this arrangement, the instrument readings are (neglecting local flow distortions such as upwash or sidewash):

$$\left.\begin{aligned} \beta_I &= \tan^{-1}(V_I/U_I) \\ \alpha_I &= \tan^{-1}(W_I/U_I) \end{aligned}\right\} \quad (56)$$

Although Equations (55) and (56) would be cumbersome for computational purposes, considerable simplification is usually possible through the neglect of small terms. For example, the orientation angles ψ_I and ϕ_I will almost invariably be equal to zero, one or more of the distances x, y, and z will usually be negligible, and the use of the small-disturbance linear velocity variables u', β', and α' will probably justify eliminating the $\tan^{-1}$ from Equations (56).

8.4 Readings of Acceleration-Measuring Instruments

As in the case of velocity-measuring instruments, acceleration-measuring instruments may have arbitrary locations and orientations with respect to the airplane. A vector equation for the linear accelerations on instrument axes may be written as:

$$\dot{\underline{V}}_I = \dot{\underline{V}} + \underline{\Omega}\times\underline{V} + \dot{\underline{\Omega}}\times\underline{r} + \underline{\Omega}\times(\underline{\Omega}\times\underline{r}) \quad (57)$$

where, in addition to the previous definitions:

$$\dot{\underline{V}}_I = \underline{l}\,\dot{U}_I + \underline{m}\,\dot{V}_I + \underline{n}\,\dot{W}_I$$

$$\dot{\underline{V}} = \underline{i}\,\dot{U} + \underline{j}\,\dot{V} + \underline{k}\,\dot{W}$$

$$\dot{\underline{\Omega}} = \underline{i}\,\dot{P} + \underline{j}\,\dot{Q} + \underline{k}\,\dot{R}$$

A comparison of Equation (57) with the vector forms of the equations of motion of the accelerometer and of the airplane itself leads to an important principle applying to acceleration-measuring instruments. This principle may be stated as follows:

> "The reading* along any direction of a linear accelerometer mounted at the airplane's center of gravity is proportional to the resultant aerodynamic force (including thrust) in that direction, and is independent of the airplane's attitude in space."

Because of the importance of this principle, it is derived from basic considerations in appendix A.

Although lengthy, the general expressions for linear accelerations along instrument axes are presented below in Cartesian form. These equations are derived from the orientation matrix $\left[L_I\right]$ used in Equation (55). Thus:

$$\begin{bmatrix}\dot{U}_I\\ \dot{V}_I\\ \dot{W}_I\end{bmatrix} = \left[L_I\right]\begin{bmatrix}\dot{U} - RV + QW - x\,(R^2+Q^2)+y(PQ-\dot{R})+z(RP+\dot{Q})\\ \dot{V} - PW + RU + x\,(PQ+\dot{R})-y(P^2+R^2)+z(QR-\dot{P})\\ \dot{W} - QU + PV + x\,(PR-\dot{Q})+y(QR+\dot{P})-z(Q^2+P^2)\end{bmatrix} \qquad (58)$$

or

$$\begin{aligned}
\dot{U}_I = {} & \left[\dot{U}-RV+QW-x\,(R^2+Q^2)+y(PQ-\dot{R})+z(RP+\dot{Q})\right]\cos\theta_I\cos\psi_I\\
& +\left[\dot{V}-PW+RU+x\,(PQ+\dot{R})-y(P^2+R^2)+z(QR-\dot{P})\right]\cos\theta_I\sin\psi_I\\
& -\left[\dot{W}-QU+PV+x\,(PR-\dot{Q})+y(QR+\dot{P})-z(Q^2+P^2)\right]\sin\theta_I\\
\dot{V}_I = {} & \left[\dot{U}-RV+QW-x\,(R^2+Q^2)+y(PQ-\dot{R})+z(RP+\dot{Q})\right](\cos\psi_I\sin\phi_I\sin\theta_I-\sin\psi_I\cos\phi_I)\\
& +\left[\dot{V}-PW+RU+x\,(PQ+\dot{R})-y(P^2+R^2)+z(QR-\dot{P})\right](\sin\psi_I\sin\phi_I\sin\theta_I+\cos\psi_I\cos\phi_I)\\
& +\left[\dot{W}-QU+PV+x\,(PR-\dot{Q})+y(QR+\dot{P})-z(Q^2+P^2)\right]\sin\phi_I\cos\theta_I
\end{aligned}$$

*Neglecting instrument lag and errors.

$$\begin{aligned}\dot{W}_I = &\left[\dot{U}-RV+QW-x\ (R^2+Q^2)+y(PQ-\dot{R})+z(RP+\dot{Q})\right] (\cos\psi_I\cos\phi_I\sin\theta_I+\sin\psi_I\sin\phi_I)\\ &+\left[\dot{V}-PW+RU+x\ (PQ+\dot{R})-y(P^2+R^2)+z(QR-\dot{P})\right] (\sin\psi_I\cos\phi_I\sin\theta_I-\cos\psi_I\sin\phi_I)\\ &+\left[\dot{W}-QU+PV+x\ (PR-\dot{Q})+y(QR+\dot{P})-z(Q^2+P^2)\right] \cos\theta_I\ \cos\phi_I\end{aligned}$$

As in the case of the linear velocities of Equations (55), the orientation angles ψ_I and ϕ_I will almost invariably be equal to

zero, and one or more of the distances x, y, and z (referred to airplane body axes) will usually be either zero or negligibly small.

Equations (57) and (58) give the linear accelerations on instrument axes, in vector and Cartesian form, respectively. To go from these equations to the readings of actual acceleration-measuring instruments requires consideration of three additional factors:

These are:

1. The effects of gravity forces on the suspended or pivoted mass (sometimes called the seismic element).
2. The dynamics of the instrument itself, which is assumed to have spring and damping forces acting on the suspended mass.
3. The effects of angular accelerations of the instrument mounting, when the accelerometer mass is pivoted as a pendulum.

The treatment used to derive the readings of actual acceleration-measuring instruments follows generally that of Reference 6. The effects of gravity forces on the suspended or pivoted mass are based on the following gravity resolution along instrument axes (using Equation(5) of Reference 3):

$$\left[g_I\right] = \left[L_I\right] \begin{bmatrix} -g \sin\theta \\ g \sin\phi \cos\theta \\ g \cos\phi \cos\theta \end{bmatrix} \qquad (59)$$

Expanding:

$$\begin{aligned} g_{X_I} = g\Big[&-\cos\theta_I \cos\psi_I \sin\theta + \cos\theta_I \sin\psi_I \sin\phi \cos\theta \\ &-\sin\theta_I \cos\phi \cos\theta\Big] \end{aligned}$$

$$\begin{aligned} g_{Y_I} = g\Big[&-(\sin\phi_I \sin\theta_I \cos\psi_I - \sin\psi_I \cos\phi_I) \sin\theta \\ &+(\sin\psi_I \sin\theta_I \sin\phi_I + \cos\psi_I \cos\phi_I) \sin\phi \cos\theta \\ &+ \sin\phi_I \cos\theta_I \cos\phi \cos\theta\Big] \end{aligned}$$

$$g_{Z_I} = g\Big[-(\cos\psi_I\cos\phi_I\sin\theta_I + \sin\psi_I\sin\phi_I)\sin\theta$$
$$+ (\sin\psi_I\cos\phi_I\sin\theta_I - \cos\psi_I\sin\phi_I)\sin\phi\cos\theta$$
$$+ \cos\phi_I\cos\theta_I\cos\phi\cos\theta\Big]$$

In the application of Equations (58) and (59), two types of linear acceleration-measuring instruments are distinguished, the linear-type accelerometer and the pendulum-type accelerometer with a rotary transducer. The sensitive direction of the linear-type accelerometer is assumed to be parallel to one of the Instrument Axes X_I, Y_I, or Z_I of Figure 6. The sensitive direction of the pendulum-type accelerometer is defined by the orientation of the support axis parallel to one of the Instrument Axes. For zero signal output, the pendulum arm, the support axis, and the sensitive direction of the accelerometer form an orthogonal system. By definition, we define the Instrument Axes for this case as being parallel to each of the orthogonal directions formed by the pendulum arm, the support axis, and the sensitive direction. The orientations of linear-and pendulum-type accelerometers relative to Instrument Axes are illustrated in Figure 7.

8.4.1 Linear-Type Accelerometer

The transfer function of a linear-type accelerometer is given by (in the notation of Reference 6):

$$\frac{\tau}{\frac{s^2 a_s}{g} - \rho} = K \quad \frac{1}{\frac{s^2}{\omega_n^2} + \frac{2}{\omega_n} s + 1} \tag{60}$$

where:

τ = Accelerometer output, or the <u>disturbed</u> motion of the seismic element relative to the case, parallel to either the X_I, Y_I or Z_I axes (Figure 7). τ has the dimensions of Length.

$s^2 a_s$ = Resultant linear acceleration of the case in the sensitive direction. $s^2 a_s$ is equal to $\dot{U}_I$, $\dot{V}_I$, or $\dot{W}_I$ of Equation (58), depending upon the sensitive direction of the instrument. $s^2 a_s$ is parallel to the direction of τ.

ρ = Tilt angle of the sensitive direction of the accelerometer to the horizontal. ρ is equal to g_{X_I}/g, g_{Y_I}/g, or g_{Z_I}/g of Equation (59), depending upon the sensitive direction of the instrument, and is normal to τ and $s^2 a_s$.

K, ω_n, ζ = Dynamic properties of the linear accelerometer.

$$K = gm'/k$$

$$\omega_n = \sqrt{k/m'}$$

$$\zeta = d/2\sqrt{m'k}$$

where m' is the seismic mass

k is the spring constant

d is the damping constant

Equation (60) assumes that the accelerometer is essentially a linear device, in the sense that the output Υ is directly proportional to the input $\frac{s^2 a_s}{g} - \rho$ at any particular frequency. This assumption is not at all inconsistent with assumptions of non-linearity for the motions of the airplane, since it is not unusual to find accelerometers with large usable linear ranges.

8.4.2 Pendulum-Type Accelerometer

A generalized transfer function for a pendulum-type accelerometer may be derived using the methods of Reference 6. Three directions are defined as in Figure 7, with unit vectors:

$\underline{s}$ = Sensitive direction

$\underline{t}$ = Normal to sensitive direction and support axis

$\underline{u}$ = Support axis

From Equation (4) of Reference 6:

$$I\,\ddot{\underline{\Upsilon}} = \underline{Q} - I\dot{\underline{\Omega}}_I - m'\,\underline{f} \times \ddot{\underline{a}} \qquad (61)$$

where:

$$\underline{\Upsilon} = 0\,\underline{s} + 0\,\underline{t} + \Upsilon\,\underline{u}$$

$$\underline{Q} = 0\,\underline{s} + 0\,\underline{t} + Q\,\underline{u}$$

$$\underline{\Omega}_I = \Omega_s\underline{s} + \Omega_t\underline{t} + \Omega_u\underline{u}$$

$$\underline{f} = -\Upsilon f\,\underline{s} + f\,\underline{t} + 0\,\underline{u}$$

$$\ddot{\underline{a}} = \dot{V}_s\,\underline{s} + \dot{V}_t\,\underline{t} + \dot{V}_u\,\underline{u}$$

The moment applied to the pendulum relative to its case Q is:

$$Q = -d\dot{T} - kT + m\,\underline{f} \times \underline{g} \tag{62}$$

where:

$$\underline{g} = g_s\,\underline{s} + g_t\,\underline{t} + g_u\,\underline{u}$$

Substituting Equation (62) into Equation (61) and taking the resultant along $\underline{u}$:

$$\frac{T}{\frac{\dot{V}_s}{g} - \frac{I}{m'fg}s\Omega_u - \frac{g_s}{g}} = K\,\frac{1}{\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n}s + 1} \tag{63}$$

where:

$$K = \frac{1}{\frac{k}{m'fg} + \frac{g_t}{g} - \frac{\dot{V}_t}{g}}$$

$$\omega_n = \sqrt{\frac{k + m'fg_t - m'f\dot{V}_t}{I}}$$

$$\zeta = \frac{d}{2\sqrt{I(k + m'fg_t - m'f\dot{V}_t)}}$$

The following definitions apply to Equation (63):

T = Accelerometer output or disturbed rotary motion of the pendulum arm about its support axis relative to the case. T is measured about the support axis $\underline{u}$, which is parallel to one of the Instrument Axes X_I, Y_I, Z_I.

$\dot{V}_s$ = Resultant linear acceleration of the case in its sensitive direction, along $\underline{s}$. $\dot{V}_s$ is equal to either $\dot{U}_I$, $\dot{V}_I$, or $\dot{W}_I$ of Equation (58) depending upon the orientation of the instrument.

Ω_u = Angular velocity of accelerometer case about the pendulum support axis direction $\underline{u}$. Ω_u is equal to either P_I, Q_I, R_I.

g_s = Component of the acceleration of gravity along the accelerometer sensitive axis $\underline{s}$. g_s is equal to either g_{X_I}, g_{Y_I}, or g_{Z_I} of Equation (59), depending on the instrument orientation.

g_t = Component of the acceleration of gravity along the normal to the sensitive direction and support axis, or along t.

$\dot{V}_t$ = Resultant linear acceleration of the case along the normal to the sensitive direction and support axis, or along t.

Equation (63) demonstrates that the response of a pendulum-type accelerometer used to measure linear acceleration (along its sensitive axis) is dependent on the linear acceleration at right angles to the sensitive direction. For example, a pendulum-type accelerometer without a spring used as a weight-downwards lateral accelerometer will have a static gain K value approximately equal to $1/n$, where n is the normal load factor. In a 2g pullup, the seismic mass will travel half as far in response to a given steady lateral acceleration as it would in straight flight. Also, the natural frequency ω_n is seen to be increased in a pullup and the damping ratio is decreased.

8.4.3 Pendulum-Type Angular Accelerometer

The transfer function of a pendulum-type accelerometer used as an angular accelerometer is:

$$\frac{\tau}{-s\Omega_u} = K \frac{1}{\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n} s + 1} \qquad (64)$$

where the notation agrees with that for the pendulum-type linear accelerometer of Section 8.4.2 except that:

$$K = I/k$$

$$\omega_n = \sqrt{k/I}$$

$$\zeta = \frac{d}{2\sqrt{k/I}}$$

9.0 REFERENCES

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APPENDIX A

THE READING OF A LINEAR ACCELEROMETER AT THE AIRPLANE'S CENTER OF GRAVITY

The important principle that the reading* along any direction of a linear accelerometer mounted at the airplane's center of gravity is proportional to the resultant aerodynamic force (including thrust) in that direction, and is independent of the airplane's attitude in space was stated in Section 8.4 of this report. This principle is derived in this Appendix from basic considerations. Letting primed values refer to the accelerometer's seismic mass, Newton's law for the rate of change of linear momentum for both the airplane and the accelerometer seismic mass is:

$$\left.\begin{aligned} \underline{F} + m\underline{g} &= m(\dot{\underline{V}} + \underline{\Omega} \times \underline{V}) \\ \underline{F}' + m'\underline{g} &= m'(\dot{\underline{V}}' + \underline{\Omega}' \times \underline{V}') \end{aligned}\right\} \quad (A\text{-}1)$$

The accelerometer is assumed to be located at the airplane's center of gravity so that the vector $\underline{r}$ of Equation (57) equals zero. Since the accelerometer's case is carried with the airplane, $\underline{\Omega}' = \underline{\Omega}$ and $\underline{V}' = \underline{V}$, if we neglect the relatively small velocities of the seismic mass relative to its case. Thus:

$$\frac{\underline{F}}{m} = \frac{\underline{F}'}{m'} \quad (A\text{-}2)$$

The vector $\underline{F}'$ represents the external forces applied to the accelerometer seismic mass. Since we are neglecting instrument lag or dynamic effects, $\underline{F}'$ arises entirely from the centering spring on the seismic mass. In the usual case this spring is linear and the accelerometer reading is proportional to $\underline{F}'$. The vector form of Equation (A-2) shows further that the accelerometer reading is proportional to the total aerodynamic force $\underline{F}$ in the direction in which the reading is made. The absence of the gravity vector from Equation (A-2) proves that the instrument reading is independent of the airplane's attitude in space.

If the accelerometer is calibrated so that its reading Υ is made equal to unity when $F'/m'g = 1.0$, then Equation (A-2) shows that the accelerometer reads directly the airplane's load factor, or aerodynamic force-to-weight ratio along its sensitive direction.

* Neglecting instrument lags and errors.

TABLE I

EQUATIONS OF AIRPLANE MOTION SUITABLE FOR FIRE-CONTROL STUDI[illegible]

	u'	β'	α'	P	Q	R	$u'Q$	$u'R$	$\beta'\alpha'$	$\beta'P$	
X	$\frac{x_{u'}}{2\tau} - s$		$\frac{x_{\alpha'}}{2\tau}$		$\frac{x_q}{2\tau} - \sin\alpha_1$						
Y		$\frac{y_{\beta'}}{2\tau} - s$		$\frac{y_p}{\tau} + \sin\alpha_1$		$\frac{y_r}{2\tau} - \cos\alpha_1$		-1.0	$\frac{y_{\beta'\alpha'}}{2\tau}$		
Z	$\frac{z_{u'}}{2\tau}$		$\frac{z_{\alpha'}}{2\tau} - s$		$\frac{z_q}{2\tau} + \cos\alpha_1$		1.0			-1.0	
L		$\frac{l_{\beta'}}{I_X}$		$\frac{l_p}{I_X} - s$	$\frac{H_e}{I_X} \sin i_T$	$\frac{l_r}{I_X} + \frac{I_{XZ}s}{I_X}$			$\frac{l_{\beta'\alpha'}}{I_X}$		
M	$\frac{m_{u'}}{I_Y}$		[illegible]	$-\frac{H_e}{I_Y} \sin i_T$	$\frac{m_q}{I_Y} - s$	$-\frac{H_e}{I_Y} \cos i_T$					
N		$\frac{n_{\beta'}}{I_Z}$		$\frac{n_p}{I_Z} + \frac{I_{XZ}s}{I_Z}$	$\frac{H_e}{I_Z} \cos i_T$	$\frac{n_r}{I_Z} - s$			$\frac{n_{\beta'\alpha'}}{I_Z}$		

* [illegible]: 1) Initial steady symmetric straight flight ([illegible] = 0)
2) Arbitrary [illegible] axes
3) Small perturbations in longitudinal velocity, sideslip, and angle of attack ([illegible] = u', [illegible] = β', [illegible]
4) Relationships between orientation angle rates and angular velocities (Equations [illegible]) computed separately

A

FIRE-CONTROL STUDIES

	θ'P	β'R	α'P	α'Q	γ'R	PQ	PP	QR	P^2	R^2	sin Φ	sin Φ sin θ	sin [illegible]
		1.0		$-1.0+\frac{x_{\dot{\alpha}'}}{2V}$							$-\frac{g}{V_1}\cos\theta_1$		
			$1.0+\frac{y_{\dot{\alpha}'}}{2V}$		$\frac{y_{\dot{\gamma}'}}{2V}$							$-\frac{g}{V_1}\sin\theta_1$	
	-1.0			$\frac{z_{\dot{\alpha}'}}{2V}$									$-\frac{g}{V_1}$
			$\frac{l_{\dot{\alpha}'}}{I_X}$		$\frac{l_{\dot{\gamma}'}}{I_X}$	$\frac{I_{XZ}}{I_X}$		$\frac{I_Y-I_Z}{I_X}$					
				$\frac{m_{\dot{\alpha}'}}{I_Y}$			$\frac{I_Z-I_X}{I_Y}$		$-\frac{I_{XZ}}{I_Y}$	$\frac{I_{XZ}}{I_Y}$			
			$\frac{n_{\dot{\alpha}'}}{I_Z}$		$\frac{n_{\dot{\gamma}'}}{I_Z}$	$\frac{I_X-I_Y}{I_Z}$		$-\frac{I_{XZ}}{I_Z}$					

[illegible])
[illegible] computed separately.

B

	P^2	R^2	sin Θ	[illegible] Φ sin Θ	sin Φ cos Θ	sin [illegible] cos Φ	cos Φ [illegible] −1	cos Φ − 1	δ_a	δ_e	δ_r	
			$-\frac{g}{V_1}\cos\Theta_1$					$-\frac{g}{V_1}\sin\Theta_1$		$\frac{X_{\delta_e}}{2\tau}$		= 0
				$-\frac{g}{V_1}\sin\Theta_1$		$\frac{g}{V_1}\cos\Theta_1$			$\frac{Y_{\delta_a}}{2\tau}$		$\frac{Y_{\delta_r}}{2\tau}$	= 0
					$-\frac{g}{V_1}\sin\Theta_1$		$\frac{g}{V_1}\cos\Theta_1$			$\frac{Z_{\delta_e}}{2\tau}$		= 0
									$\frac{L_{\delta_a}}{I_X}$		$\frac{L_{\delta_r}}{I_X}$	= 0
	$-\frac{I_{-}}{I_Y}$	$\frac{L_Z}{I_Y}$								$\frac{M_{\delta_e}}{I_Y}$		= 0
									$\frac{N_{\delta_a}}{I_Z}$		$\frac{N_{\delta_r}}{I_Z}$	= 0

TABLE II

EQUATIONS OF AIRPLANE MOTION SUITABLE FOR CALCULATION OF

	β'	α'	P	q	R	$\beta'\alpha'$	$\beta'P$	$\alpha'P$
Y	$\frac{y_{\beta'}}{2\tau} - s$		$\frac{y_{\dot\phi}}{2\tau} + \sin\eta_1$		$\frac{y_{\dot\psi}}{2\tau} - \cos\eta_1$	$\frac{y_{\beta'\alpha'}}{2\tau}$		$1.0 + \frac{y_{\dot\phi\alpha'}}{2\tau}$
Z		$\frac{z_{\alpha'}}{2\tau} - s$		$\frac{z_{\dot\theta}}{2\tau} + \cos\eta_1$			-1.0	
L	$\frac{l_{\beta'}}{CI_X}$	$\frac{l_{\alpha'}}{CI_X}$	$\frac{l_{\dot\phi}}{CI_X} - s$	$\frac{H_e}{I_X}\sin i_T$	$\frac{l_{\dot\psi}}{CI_X} + Q_1\left(\frac{I_Y - I_Z}{I_X}\right)$	$\frac{l_{\beta'\alpha'}}{CI_X}$		$\frac{l_{\dot\phi\alpha'}}{CI_X}$
M		$\frac{m_{\dot\alpha'}s + m_{\alpha'}}{CI_Y}$	$-\frac{H_e}{I_Y}\sin i_T$	$\frac{m_{\dot\theta}}{CI_Y} - s$	$-\frac{H_e}{I_Y}\cos i_T$			
N	$\frac{n_{\beta'}}{CI_Z}$	$\frac{n_{\alpha'}}{CI_Z}$	$\frac{n_{\dot\phi}}{CI_Z} + Q_1\left(\frac{I_X - I_Y}{I_Z}\right)$	$\frac{H_e}{I_Z}\cos i_T$	$\frac{n_{\dot\psi}}{CI_Z} - s$	$\frac{n_{\beta'\alpha'}}{CI_Z}$		$\frac{n_{\dot\phi\alpha'}}{CI_Z}$

* ASSUMPTIONS:
1) Principal axes ($I_{XZ} = 0$)
2) Constant Longitudinal velocity ($u' = 0$)
3) Gravity terms simplified ($\theta = 0$)
4) $\phi = \int p \, dt$
5) Initial symmetric flight ($\beta_1 \; P_1 \; R_1 \; \phi_1$ ALL $= 0$)
6) Small perturbations in sideslip and angle of attack ($\Delta\beta = \beta'$, $\Delta\alpha = \alpha'$)

II

CALCULATION OF RAPID ROLLING MANEUVERS*

P	α'P	α'q	α'R	Pq	PR	qR	$\sin\phi$	$\cos\phi - 1$	
	$1.0+\frac{y_{\dot{\phi}\alpha'}}{2T}$		$\frac{y_{\dot{\psi}\alpha'}}{2T}$				$\frac{g}{V_1}$		$\frac{y}{2}$
0		$\frac{z_{\dot{\theta}\alpha'}}{2T}$						$\frac{g}{V_1}$	
	$\frac{l_{\dot{\phi}\alpha'}}{C_{I_X}}$		$\frac{l_{\dot{\psi}\alpha'}}{C_{I_X}}$			$\frac{I_Y - I_Z}{I_X}$			$\frac{l}{C_I}$
		$\frac{m_{\dot{\theta}\alpha'}}{C_{I_Y}}$			$\frac{I_Z - I_Y}{I_Y}$				
	$\frac{n_{\dot{\phi}\alpha'}}{C_{I_Z}}$		$\frac{n_{\dot{\psi}\alpha'}}{C_{I_Z}}$	$\frac{I_X - I_Y}{I_Z}$					$\frac{n}{C_I}$

$\beta', \Delta\alpha = \alpha'$)

B

qR	sin φ	cos φ -1	δ_a	δ_e	δ_r	
	$\frac{g}{V_1}$		$\frac{y_{\delta_a}}{2\tau}$		$\frac{y_{\delta_r}}{2\tau}$	= 0
		$\frac{g}{V_1}$		$\frac{z_{\delta_e}}{2\tau}$		= 0
$\frac{I_Y - I_Z}{I_X}$			$\frac{l_{\delta_a}}{C_{I_X}}$		$\frac{l_{\delta_r}}{C_{I_X}}$	= 0
				$\frac{m_{\delta_e}}{C_{I_Y}}$		= 0
			$\frac{n_{\delta_a}}{C_{I_Z}}$		$\frac{n_{\delta_r}}{C_{I_Z}}$	= 0

C

TABLE III

LONGITUDINAL VELOCITY DERIVATIVES ON BODY AXES

(C_L, C_D and C_m are assumed to include all running-engine effects except for the net thrust)

DERIVATIVE	SYMBOL	STABILITY AXES EQUIVALENT
$\frac{\partial X}{\partial u'} \frac{1}{q_1 S}$	$x_{u'}$	$(-2C_{D_1} - M_1 C_{D_M} - 2q_1 C_{D_q}) \cos^2\alpha_1 + (-C_{L_\alpha} - C_{D_1}) \sin^2\alpha_1$ $+ (C_{L_1} + C_{D_\alpha} + M_1 C_{L_M} + 2q_1 C_{L_q}) \cos\alpha_1 \sin\alpha_1 + \frac{2T}{m} T_V \cos i_T \cos\alpha_1$
$\frac{\partial Z}{\partial u'} \frac{1}{q_1 S}$	$z_{u'}$	$(-2C_{L_1} - M_1 C_{L_M} - 2q_1 C_{L_q}) \cos^2\alpha_1 + (C_{D_\alpha} - C_{L_1}) \sin^2\alpha_1$ $+ (C_{L_\alpha} - C_{D_1} - M_1 C_{D_M} - 2q_1 C_{D_q}) \cos\alpha_1 \sin\alpha_1 - \frac{2T}{m} T_V \sin i_T \cos\alpha_1$
$\frac{\partial L}{\partial u'} \frac{1}{q_1 S b}$	$l_{u'}$	$(2C_{l_1} + M_1 C_{l_M} + 2q_1 C_{l_q}) \cos\alpha_1$
$\frac{\partial M}{\partial u'} \frac{1}{q_1 S c}$	$m_{u'}$	$(M_1 C_{m_M} + 2q_1 C_{m_q} + \frac{2T}{m} T_V \frac{z_T}{c} - 2T'_{c_1} \frac{z_T}{c}) \cos\alpha_1$ $- C_{m_\alpha} \sin\alpha_1$
$\frac{\partial N}{\partial u'} \frac{1}{q_1 S b}$	$n_{u'}$	$(2C_{n_1} + M_1 C_{n_M} + 2q_1 C_{n_q}) \cos\alpha_1$

TABLE IV

ANGLE OF ATTACK DERIVATIVES ON BODY AXES

(C_L, C_D, and C_m are assumed to include all running-engine effects, except for the net thrust)

DERIVATIVE	SYMBOL	STABILITY AXES EQUIVALENT
$\frac{\partial X}{\partial \alpha'} \frac{1}{q_1 S}$	$x_{\alpha'}$	$(C_{L_1} - C_{D_\alpha}) \cos^2\alpha_1 + (C_{L_\alpha} - C_{D_1} - M_1 C_{D_M} - 2q_1 C_{D_q}) \cos\alpha_1 \sin\alpha_1$ $+ (2C_{L_1} + M_1 C_{L_M} + 2q_1 C_{L_q}) \sin^2\alpha_1 + \frac{2T}{m} T_V \cos i_T \sin\alpha_1$
$\frac{\partial Z}{\partial \alpha'} \frac{1}{q_1 S}$	$z_{\alpha'}$	$(-C_{L_\alpha} - C_{D_1}) \cos^2\alpha_1 + (-C_{L_1} - C_{D_\alpha} - M_1 C_{L_M} - 2q_1 C_{L_q}) \cos\alpha_1 \sin\alpha_1$ $+ (-2C_{D_1} - M_1 C_{D_M} - 2q_1 C_{D_q}) \sin^2\alpha_1 - \frac{2T}{m} T_V \sin i_T \sin\alpha_1$
$\frac{\partial L}{\partial \alpha'} \frac{1}{q_1 Sb}$	$\ell_{\alpha'}$	$(2C_{\ell_1} + M_1 C_{\ell_M} + 2q_1 C_{\ell_q}) \sin\alpha_1$
$\frac{\partial M}{\partial \alpha'} \frac{1}{q_1 Sc}$	$m_{\alpha'}$	$C_{m_\alpha} \cos\alpha_1 + (M_1 C_{m_M} + 2q_1 C_{m_q} + \frac{2T}{m} T_V \frac{z_T}{c} - 2T_{c_1}' \frac{z_T}{c}) \sin\alpha_1$
$\frac{\partial M}{\partial \dot{\alpha}'} \frac{1}{q_1 Sc}$	$m_{\dot{\alpha}'}$	$C_{m_{\dot{\alpha}}}$
$\frac{\partial N}{\partial \alpha'} \frac{1}{q_1 Sb}$	$n_{\alpha'}$	$(2C_{n_1} + M_1 C_{n_M} + 2q_1 C_{n_q}) \sin\alpha_1$

TABLE V

SIDESLIP DERIVATIVES ON BODY AXES

DERIVATIVE	SYMBOL	STABILITY AXES EQUIVALENT
$\frac{\partial Y}{\partial \beta'} \frac{1}{q_1 S}$	$y_{\beta'}$	C_{Y_β}
$\frac{\partial}{\partial \alpha'}\left(\frac{\partial Y}{\partial \beta'}\right)\frac{1}{q_1 S}$	$y_{\beta'\alpha'}$	$C_{Y_{\beta\alpha}}$
$\frac{\partial L}{\partial \beta'} \frac{1}{q_1 S b}$	$l_{\beta'}$	$C_{l_\beta} \cos \alpha_1 - C_{n_\beta} \sin \alpha_1$
$\frac{\partial}{\partial \alpha'}\left(\frac{\partial L}{\partial \beta'}\right)\frac{1}{q_1 S b}$	$l_{\beta'\alpha'}$	$(C_{l_{\beta\alpha}} - C_{n_\beta}) \cos \alpha_1 - (C_{n_{\beta\alpha}} + C_{l_\beta}) \sin \alpha_1$
$\frac{\partial N}{\partial \beta'} \frac{1}{q_1 S b}$	$n_{\beta'}$	$C_{n_\beta} \cos \alpha_1 + C_{l_\beta} \sin \alpha_1$
$\frac{\partial}{\partial \alpha'}\left(\frac{\partial N}{\partial \beta'}\right)\frac{1}{q_1 S b}$	$n_{\beta'\alpha'}$	$(C_{n_{\beta\alpha}} + C_{l_\beta}) \cos \alpha_1 + (C_{l_{\beta\alpha}} - C_{n_\beta}) \sin \alpha_1$

Note: The sideslip rate derivatives like $y_{\dot{\beta}'}$ are identical in form to the corresponding sideslip derivatives.

TABLE VI

SYMMETRIC ROTARY DERIVATIVES ON BODY AXES

DERIVATIVE	SYMBOL	STABILITY AXES EQUIVALENT
$\frac{\partial X}{\partial q}\ \frac{1}{q_1 S}$	$x_{\dot\theta}$	$-C_{D_{\dot\theta}} \cos \alpha_1 + C_{L_{\dot\theta}} \sin \alpha_1$
$\frac{\partial}{\partial \alpha}\left(\frac{\partial X}{\partial q}\right)\frac{1}{q_1 S}$	$x_{\dot\theta\alpha'}$	$(-C_{D_{\dot\theta\alpha}} + C_{L_{\dot\theta}}) \cos \alpha_1 + (C_{L_{\dot\theta\alpha}} + C_{D_{\dot\theta}}) \sin \alpha_1$
$\frac{\partial Z}{\partial q}\ \frac{1}{q_1 S}$	$z_{\dot\theta}$	$-C_{L_{\dot\theta}} \cos \alpha_1 - C_{D_{\dot\theta}} \sin \alpha_1$
$\frac{\partial}{\partial \alpha'}\left(\frac{\partial Z}{\partial q}\right)\frac{1}{q_1 S}$	$z_{\dot\theta\alpha'}$	$-(C_{L_{\dot\theta\alpha}} + C_{D_{\dot\theta}}) \cos \alpha_1 - (C_{D_{\dot\theta\alpha}} - C_{L_{\dot\theta}}) \sin \alpha_1$
$\frac{\partial M}{\partial q}\ \frac{1}{q_1 S c}$	$m_{\dot\theta}$	$C_{m_{\dot\theta}}$
$\frac{\partial}{\partial \alpha'}\left(\frac{\partial M}{\partial q}\right)\frac{1}{q_1 S c}$	$m_{\dot\theta\alpha'}$	$C_{m_{\dot\theta\alpha}}$

DERIVATIVE	SYMBOL	ST... COEF...		
		cos α_1	sin α_1	cos
$\frac{\partial Y}{\partial \dot{p}} \frac{1}{q_1 S}$	$y_{\dot{\phi}}$	$C_{Y_{\dot{\phi}}}$	$-C_{Y_{\dot{\psi}}}$	
$\frac{\partial}{\partial \alpha}\left(\frac{\partial Y}{\partial \dot{p}}\right) \frac{1}{q_1 S}$	$y_{\dot{\phi}\alpha'}$	$C_{Y_{\dot{\phi}\alpha}} - C_{Y_{\dot{\psi}}}$	$-C_{Y_{\dot{\psi}\alpha}} - C_{Y_{\dot{\phi}}}$	
$\frac{\partial L}{\partial \dot{p}} \frac{1}{q_1 S b}$	$l_{\dot{\phi}}$			$-C_{l_{\dot{\psi}}}$ - (
$\frac{\partial}{\partial \alpha}\left(\frac{\partial L}{\partial \dot{p}}\right) \frac{1}{q_1 S b}$	$l_{\dot{\phi}\alpha'}$			$-C_{l_{\dot{\psi}\alpha}}$ - C $-2C_{l_{\dot{\phi}}}$ + 2
$\frac{\partial N}{\partial \dot{p}} \frac{1}{q_1 S b}$	$n_{\dot{\phi}}$			$C_{l_{\dot{\phi}}}$ - C
$\frac{\partial}{\partial \alpha}\left(\frac{\partial N}{\partial \dot{p}}\right) \frac{1}{q_1 S b}$	$n_{\dot{\phi}\alpha'}$			$C_{l_{\dot{\phi}\alpha}}$ - C $-2C_{n_{\dot{\phi}}}$ - 2C
$\frac{\partial Y}{\partial \dot{r}} \frac{1}{q_1 S}$	$y_{\dot{\psi}}$	$C_{Y_{\dot{\psi}}}$	$C_{Y_{\dot{\phi}}}$	
$\frac{\partial}{\partial \alpha}\left(\frac{\partial Y}{\partial \dot{r}}\right) \frac{1}{q_1 S}$	$y_{\dot{\psi}\alpha'}$	$C_{Y_{\dot{\psi}\alpha}} + C_{Y_{\dot{\phi}}}$	$C_{Y_{\dot{\phi}\alpha}} - C_{Y_{\dot{\psi}}}$	
$\frac{\partial L}{\partial \dot{r}} \frac{1}{q_1 S b}$	$l_{\dot{\psi}}$			$C_{l_{\dot{\phi}}}$ - (
$\frac{\partial}{\partial \alpha}\left(\frac{\partial L}{\partial \dot{r}}\right) \frac{1}{q_1 S b}$	$l_{\dot{\psi}\alpha'}$			$C_{l_{\dot{\phi}\alpha}} - C_{n_{\dot{\psi}}}$ $-2C_{l_{\dot{\psi}}} - 2C_{n}$
$\frac{\partial N}{\partial \dot{r}} \frac{1}{q_1 S b}$	$n_{\dot{\psi}}$			$C_{l_{\dot{\psi}}} + C_{n_{\dot{\phi}}}$
$\frac{\partial}{\partial \alpha}\left(\frac{\partial N}{\partial \dot{r}}\right) \frac{1}{q_1 S b}$	$n_{\dot{\psi}\alpha'}$			$C_{l_{\dot{\psi}\alpha}}$ + C $-2C_{n_{\dot{\psi}}}$ + 2 C

TABLE VII

TRIC ROTARY DERIVATIVES ON BODY AXES

STABILITY AXES EQUIVALENT		
COEFFICIENT OF $\cos\alpha_1 \sin\alpha_1$	$\cos^2\alpha_1$	$\sin^2\alpha_1$
$l_{\dot\psi} - C_{n_{\dot\phi}}$	$C_{l_{\dot\phi}}$	$C_{n_{\dot\psi}}$
$_{\dot\psi\alpha} - C_{n_{\dot\phi\alpha}}$ $_{\dot\phi} + 2C_{n_{\dot\psi}}$	$C_{l_{\dot\phi\alpha}} - C_{l_{\dot\psi}} - C_{n_{\dot\phi}}$	$C_{n_{\dot\psi\alpha}} + C_{l_{\dot\psi}} + C_{n_{\dot\phi}}$
$C_{l_{\dot\phi}} - C_{n_{\dot\psi}}$	$C_{n_{\dot\phi}}$	$-C_{l_{\dot\psi}}$
$x - C_{n_{\dot\psi\alpha}}$ $- 2C_{l_{\dot\psi}}$	$C_{n_{\dot\phi\alpha}} + C_{l_{\dot\phi}} - C_{n_{\dot\psi}}$	$-C_{l_{\dot\psi\alpha}} - C_{l_{\dot\phi}} + C_{n_{\dot\psi}}$
$C_{l_{\dot\phi}} - C_{n_{\dot\psi}}$	$C_{l_{\dot\psi}}$	$-C_{n_{\dot\phi}}$
$_{\dot\phi\alpha} - C_{n_{\dot\psi\alpha}}$ $l_{\dot\psi} - 2C_{n_{\dot\phi}}$	$C_{l_{\dot\psi\alpha}} + C_{l_{\dot\phi}} - C_{n_{\dot\psi}}$	$-C_{n_{\dot\phi\alpha}} - C_{l_{\dot\phi}} + C_{n_{\dot\psi}}$
$l_{\dot\psi} + C_{n_{\dot\phi}}$	$C_{n_{\dot\psi}}$	$C_{l_{\dot\phi}}$
$_{\dot\psi\alpha} + C_{n_{\dot\phi\alpha}}$ $n_{\dot\psi} + 2C_{l_{\dot\phi}}$	$C_{n_{\dot\psi\alpha}} + C_{l_{\dot\psi}} + C_{n_{\dot\phi}}$	$C_{l_{\dot\phi\alpha}} - C_{l_{\dot\psi}} - C_{n_{\dot\phi}}$

B

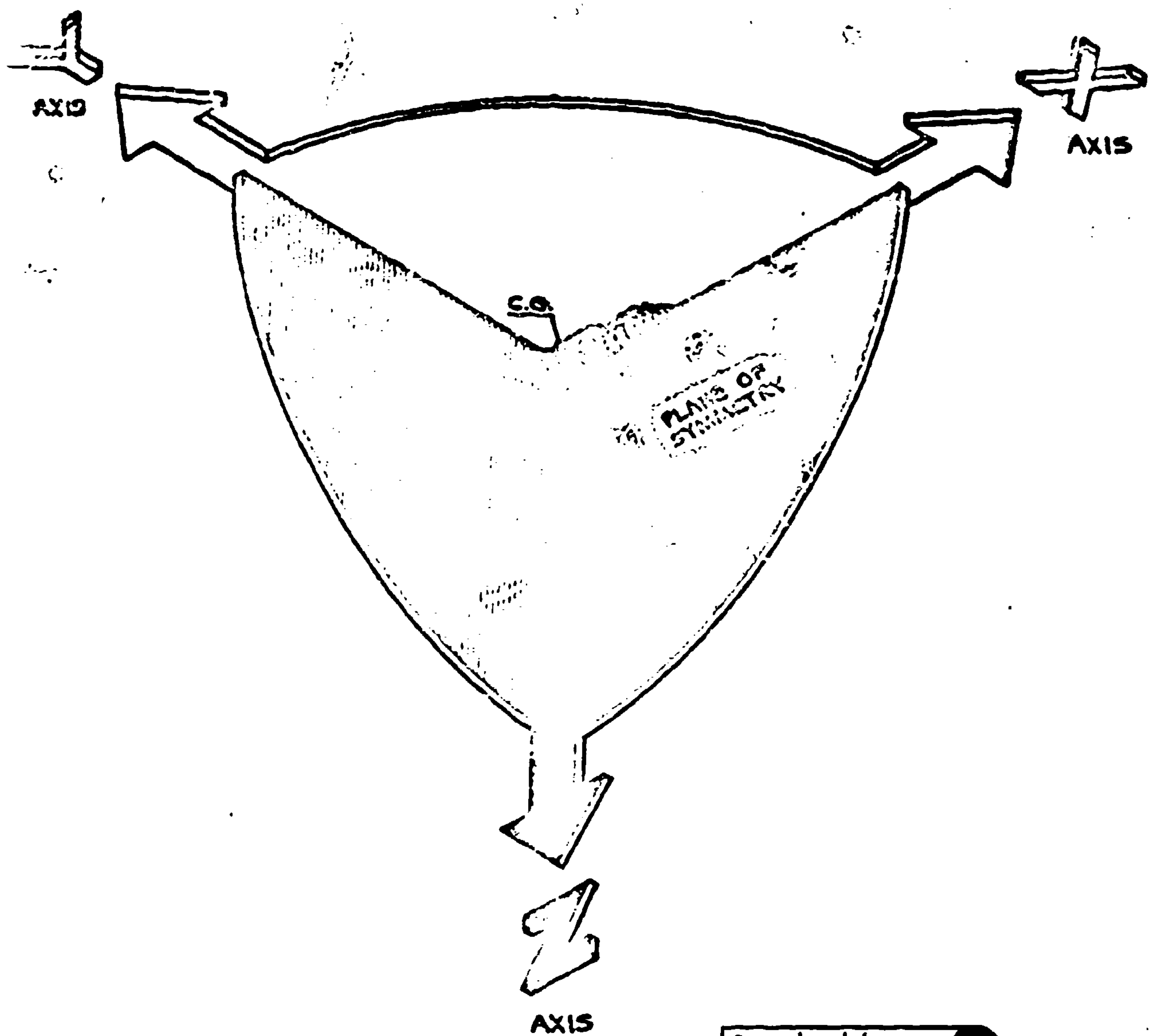
TABLE VIII

CONTROL SURFACE DERIVATIVES ON BODY AXES

(δ is a general control surface perturbation)

DERIVATIVE	SYMBOL	STABILITY AXES EQUIVALENT
$\frac{\partial X}{\partial \delta} \frac{1}{q_1 S}$	x_δ	$-C_{D_\delta} \cos \alpha_1 + C_{L_\delta} \sin \alpha_1$
$\frac{\partial Y}{\partial \delta} \frac{1}{q_1 S}$	y_δ	C_{Y_δ}
$\frac{\partial Z}{\partial \delta} \frac{1}{q_1 S}$	z_δ	$-C_{L_\delta} \cos \alpha_1 - C_{D_\delta} \sin \alpha_1$
$\frac{\partial L}{\partial \delta} \frac{1}{q_1 Sb}$	l_δ	$C_{l_\delta} \cos \alpha_1 - C_{n_\delta} \sin \alpha_1$
$\frac{\partial M}{\partial \delta} \frac{1}{q_1 Sc}$	m_δ	C_{m_δ}
$\frac{\partial N}{\partial \delta} \frac{1}{q_1 Sb}$	n_δ	$C_{n_\delta} \cos \alpha_1 + C_{l_\delta} \sin \alpha_1$

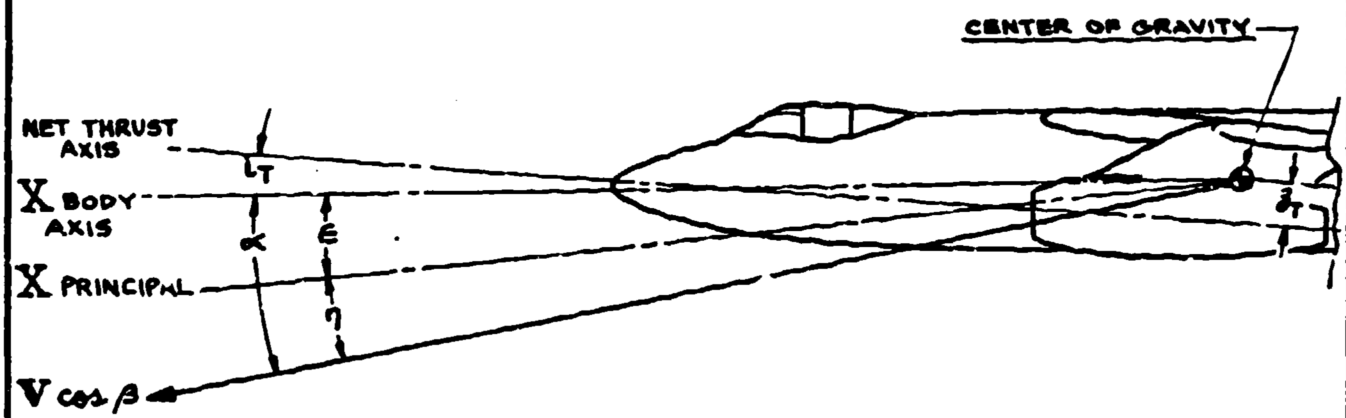
AIRPLANE BODY AXES



FORM 30-250
30-1-47

DOUGLAS AIRCRAFT COMPANY, INC. EL SEGUNDO PLANT EL SEGUNDO, CALIFORNIA

ANGULAR RELATIONSHIPS IN PLANE OF SYMMETRY

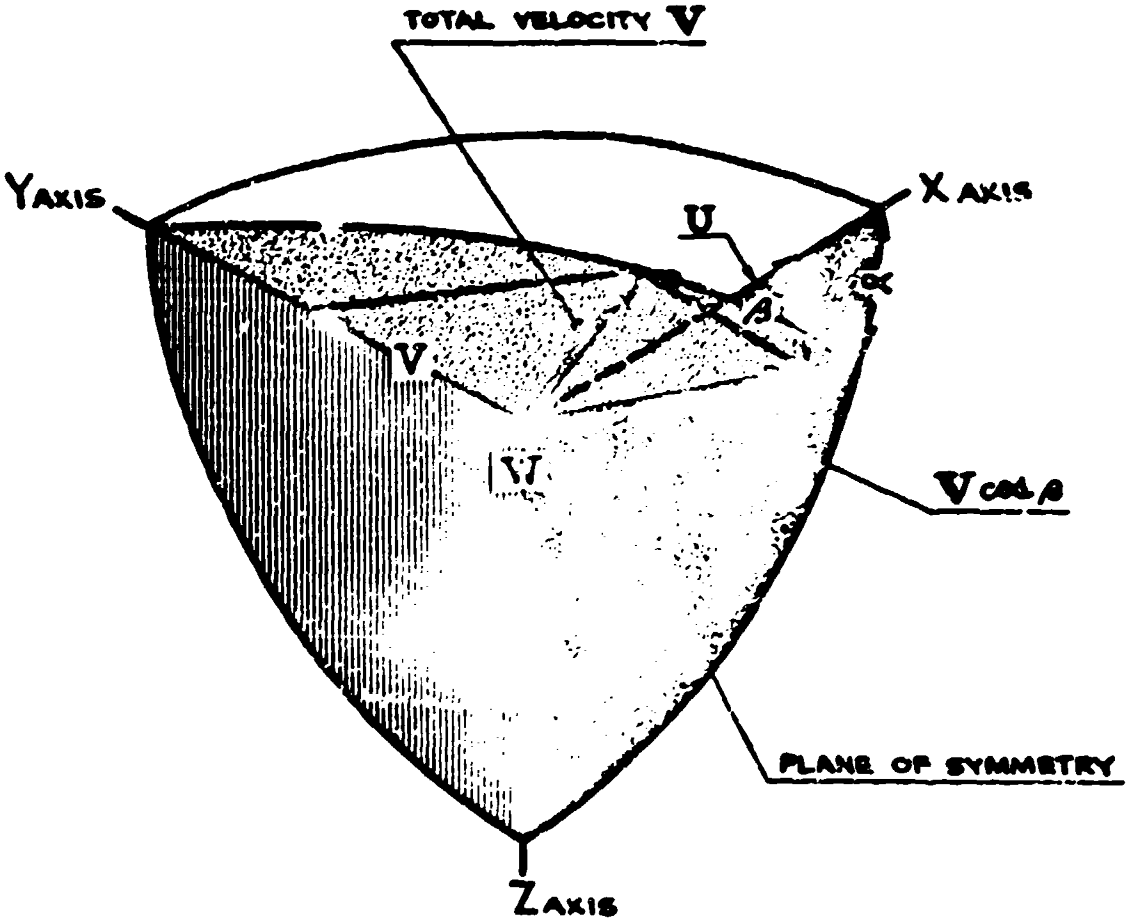


Form 30-250
10-1-47

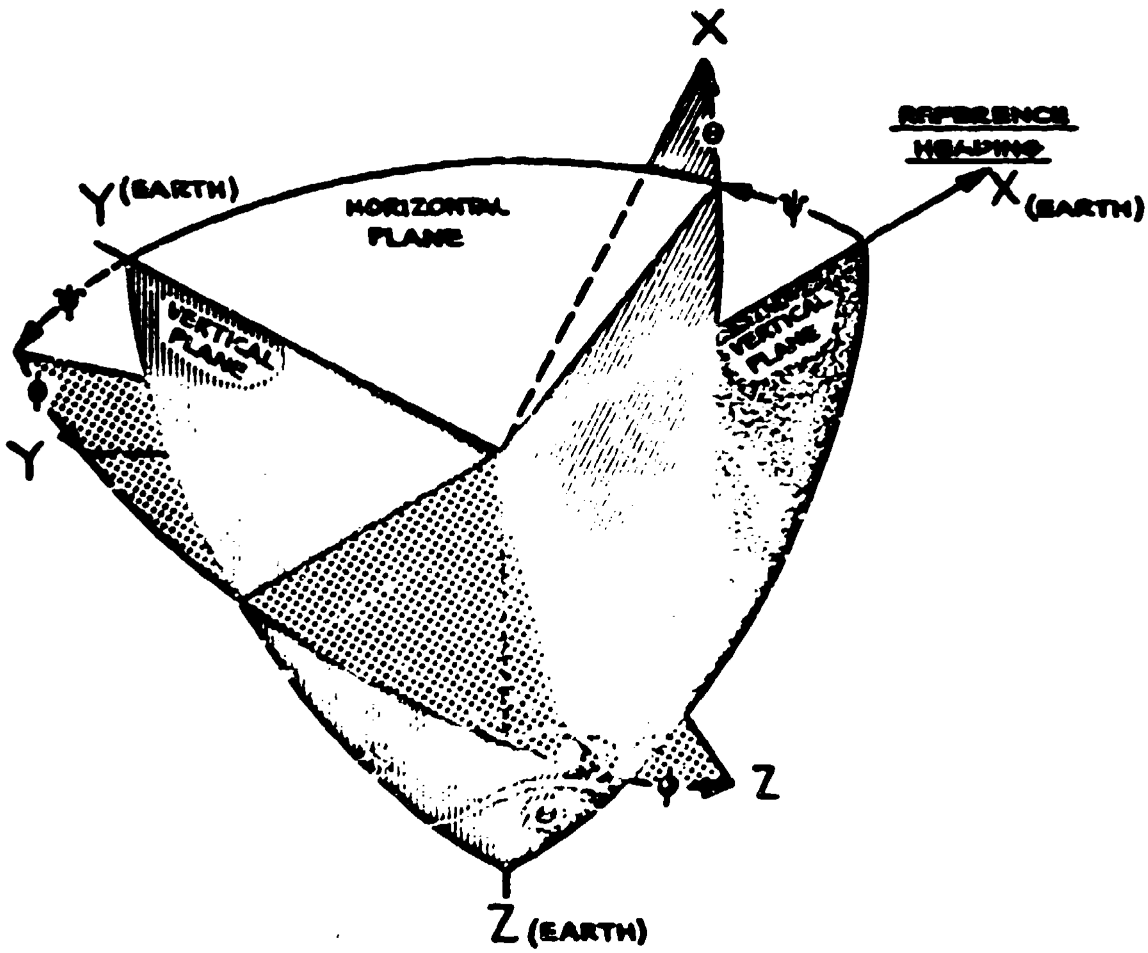
DOUGLAS AIRCRAFT COMPANY, INC. EL SEGUNDO PLANT EL SEGUNDO, CALIFORNIA

ANGLES OF SIDESLIP AND ATTACK

$$U = V \cos\beta \cos\alpha \qquad \beta = \sin^{-1} v/V$$
$$v = V \sin\beta \qquad \alpha = \tan^{-1} W/U$$
$$W = V \cos\beta \sin\alpha$$



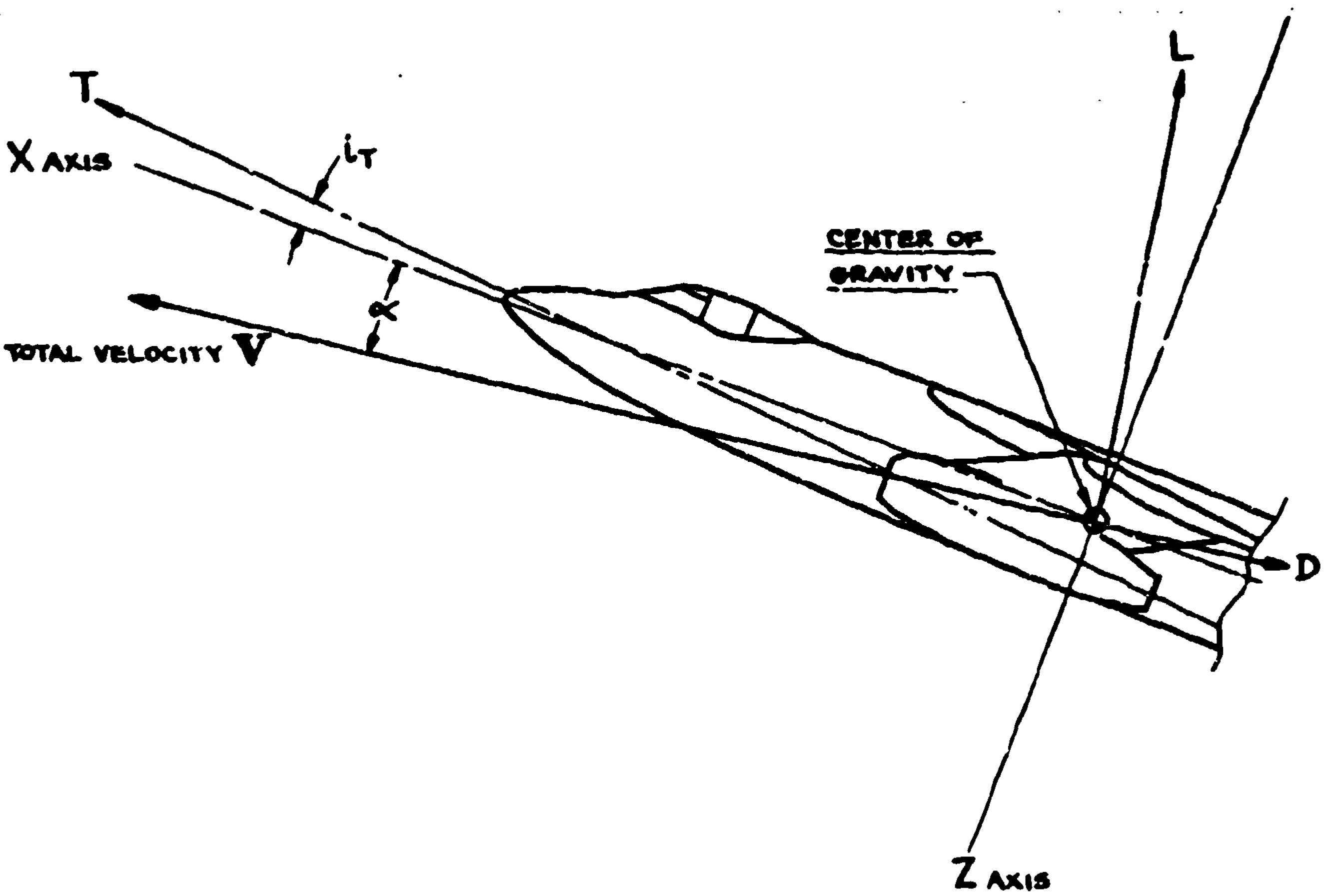
ORIENTATION ANGLES OF BODY AXES



ORDER OF ROTATIONS

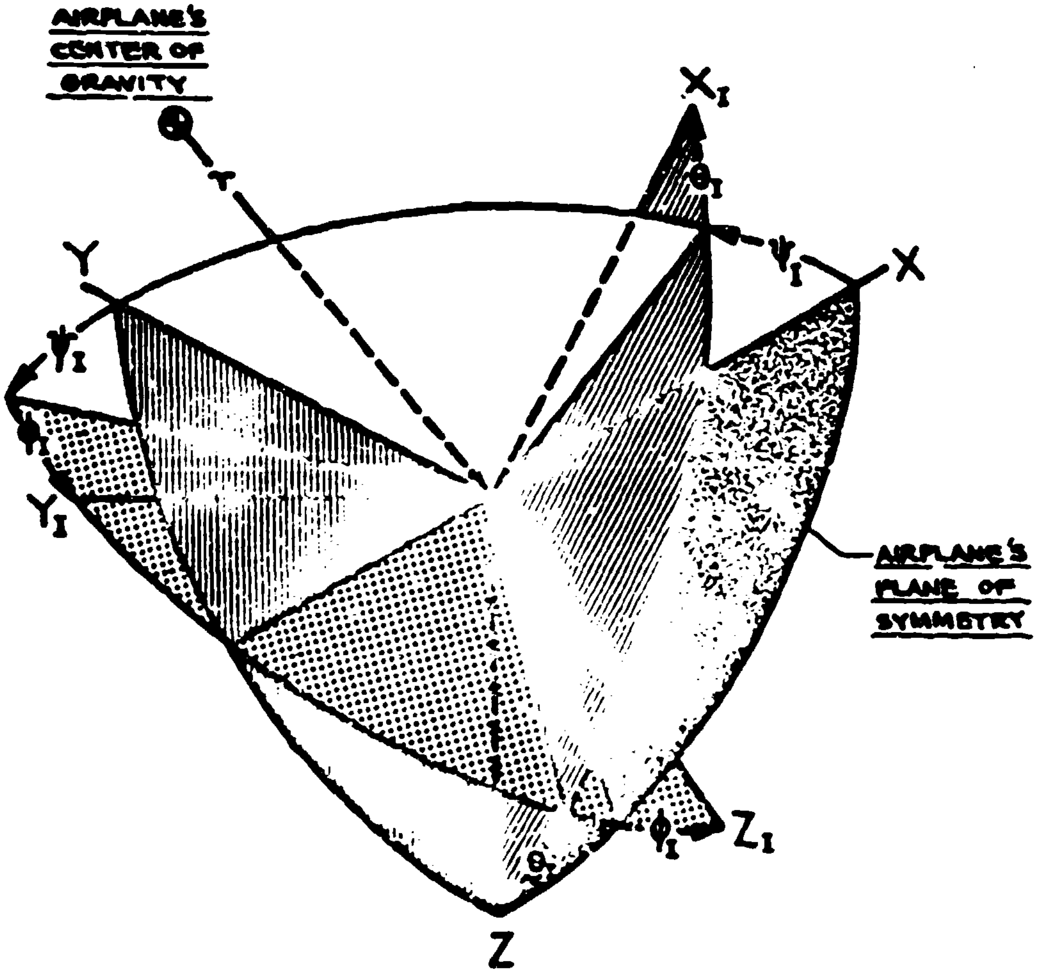
ψ , θ , φ

AERODYNAMIC FORCE RESOLUTIONS



$$X = L \sin \alpha - D \cos \alpha + T \cos i_T$$
$$Z = -L \cos \alpha - D \sin \alpha - T \sin i_T$$

GENERAL INSTRUMENT AXES



AIRPLANE BODY AXES

X, Y, Z

ORDER OF ROTATIONS

ψ_I, θ_I, ϕ_I

INSTRUMENT AXES

X_I, Y_I, Z_I

DOUGLAS AIRCRAFT COMPANY, INC. EL SEGUNDO PLANT EL SEGUNDO, CALIFORNIA

ACCELEROMETER ORIENTATIONS RELATIVE TO INSTRUMENT AXES

NOTE: EACH OF THE THREE DIRECTIONS s, t, AND u COINCIDE WITH ONE OF THE INSTRUMENT AXES X_I, Y_I, Z_I